# $C^\infty$-normal forms of local vector fields

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July 4, 2001

## 1 Introduction

Let us recall that a *vector field* (v.f.) on a smooth manifold $M$ is a mapping

$$v : M \to T_*M$$

into the tangent bundle of $M$. In local coordinates $v(x) = \sum v_i(x)\frac{\partial}{\partial x^i}$, where $\frac{\partial}{\partial x^i}$ is a basis in $T_xM$.

Every vector field $v$ defines the differential equation

$$\frac{dx}{dt} = v(x).$$

A *solution (or a trajectory of* $v$ *)* is a smooth curve $x : \mathbb{R}^1 \to M$ satisfying the equation, i.e.

$$\frac{dx}{dt} = v(x(t)).$$

The main problem is an investigation of solutions.

A diffeomorphism $\Phi : M \to M$ transforms the vector field $v$ into the vector field

$$(\Phi_*v)(x) := (\Phi^{'}(x))^{-1}v(\Phi(x)) \quad (x \in M)$$

where $\Phi^{'}(x)$ is the Jacobi matrix of the mapping $\Phi(x)$.

Two v.f. $v$ and $\tilde{v}$ are said to be *conjugate* if the there exists a transformation $\Phi$ such that

$$\tilde{v} = \Phi_*v.$$

The natural idea is to carry the v.f. $v$ by a transformation into a simple "normal" form more convenient for the investigation. The problem then arises of v.f. classification up to conjugacy. This is a very complicate problem and not many general results are known up to now.

A part of the mentioned "global" problem is its local setting.

Namely, two vector fields $v$ and $\tilde{v}$ are called *locally conjugate* in a neighborhood of a point $x_0 \in M$ if there exists a local transformation

$$\Phi : (U, x_0) \to (V, x_0)$$

of a neighborhood $U$ to a neighborhood $V$ such that

$$\tilde{v}(x) = (\Phi_* v)(x), \quad x \in U.$$

In the local context, one can identify a neighborhood $U$ and the tangent space $T_x M$ $(x \in U)$ with $\mathbb{R}^n$. After this identification a local vector field $v = \sum\limits_i v_i(x) \frac{\partial}{\partial x^i}$ may be identified with the mapping $v : (\mathbb{R}^n, x_0) \to \mathbb{R}^n$.

In contrast to the global problem there is a lot of local investigations and results starting from Poincaré.

**Example 1.1.** Let $v(x_0) = e \neq 0$. Then according to the "Main Theorem of Differential Equations" (see [1]) the vector field $v$ may be locally "rectified" i.e. it is locally conjugate with the constant v.f.

$$(\Phi_* v)(x) = e, \quad x \in U.$$

Obviously, trajectories of the constant vector field are straight lines

$$x(t) = te + x.$$

The smoothness class of the transformation $\Phi$ in Example 1.1 is the same as that of $v$.

*In what follows we consider $C^\infty$-v.f. and $C^\infty$-transformations $\Phi$. Invertibility of $\Phi$ in a neighborhood of $x_0$ is equivalent to the invertibility of the Jacobi matrix $\Phi'(x_0)$.*

A point $x_0 \in M$ such that $v(x_0) \neq 0$ is called *nonsingular* for the v.f. $v$. The above mentioned Theorem gives us a structure of a vector field in a neighborhood of a nonsingular point.

**Example 1.2.** Let

$$v(x) = \lambda x + h(x), \quad x \in \mathbb{R}^1, \ h(0) = h'(0) = 0$$

be a $C^\infty$-vector field on the line $\mathbb{R}^1$ in a neighborhood of the origin, $x_0 = 0$. This point is singular. But *if $\lambda \neq 0$ then $v$ is locally $C^\infty$-conjugate with the linear v.f. $\tilde{v}(x) = \lambda x$.*

Indeed, let us look for the transformation of the form

$$\Phi(x) = x + x\varphi(x), \quad \varphi(0) = 0.$$

Then we get the equation

$$(1 + \varphi + x\varphi^{'}(x))^{-1}(\lambda x + \lambda x\varphi(x) + h(x + x\varphi(x))) = \lambda x,$$

or, after a simple transformation,

$$\varphi^{'}(x) = \frac{h(x + x\varphi(x))}{\lambda x^2}.$$

Since $h(0) = h^{'}(0) = 0$, the function $h$ may be rewritten in the form

$$h(x) = x^2 g(x),$$

where $g$ is a $C^\infty$-function in a neighborhood of zero. Thus, we get the Cauchy problem

$$\varphi^{'}(x) = (1 + \varphi(x))^2 \frac{g(x + \varphi(x))}{\lambda}, \quad \varphi(0) = 0.$$

By the Existence Theorem, it has a local $C^\infty$-solution.

Note that trajectories of the linear v.f. $v(x) = \lambda x$ have the form

$$x(t) = e^{\lambda t} x.$$

# 2 Vector fields on the line

Continuing Example 1.2, let us consider further degenerations of $v$, when $\lambda = 0$. Assume that

$$v(x) = \alpha x^r + h(x), \quad \alpha \neq 0,\ h(0) = \ldots = h^{(r)}(0) = 0, \quad r \geq 2.$$

**Theorem 2.1.** *There exists a local $C^\infty$-transformation $\Phi(x)$, bringing $v$ to the normal form*

$$\tilde{v}(x) = \pm x^r + dx^{2r-1}.$$

The sign $\pm$ depends on $r$ : if $r$ is odd then the sign is "+". Otherwise $\pm$ = sign $\alpha$.

To prove Theorem 2.1 we start from

**Lemma 2.1.** *Let*

$$w(x) = \tilde{v}(x) + x^{2r} g(x) \tag{2.1}$$

*with a local $C^\infty$-function $g$. Then $w$ and $\tilde{v}$ are locally conjugate.*

**Proof.** Setting $\Phi(x) = x + x^r\varphi(x)$, $\varphi(0) = 0$, we obtain the equation

$$(1 + x^r\varphi'(x) + rx^{r-1}\varphi(x))\tilde{v}(x) =$$
$$= \tilde{v}(\Phi(x)) + (\Phi(x))^{2r}g(\Phi(x)).$$

After simple transformations we obtain the equation

$$\varphi'(x) = h(x,\varphi(x))$$

where $h(x,y)$ is a $C^\infty$-function. By the Existence Theorem there is a local $C^\infty$-solution $\varphi, \varphi(0) = 0$. □

To complete the proof of Theorem 2.1 it remains to check that $v$ may be transformed to the form (2.1).

To this end assume that

$$v(x) = \alpha x^r + ax^q + o(x^q)$$

with $q \in \overline{r+1,\, 2r-2}$ and show that by a polynomial transformation of the form

$$\Phi(x) = x + cx^{q-r+1}$$

one can "kill" the summand $ax^q$.

Indeed,

$$(\Phi_* v)(x) = (1+(q-r+1)cx^{q-r})^{-1}\left[\alpha(x + cx^{q-r+1})^r + a(x + cx^{q-r+1})^q + o(x^q)\right] =$$
$$= \alpha x^r + [\alpha c(2r-q-1) + a]x^q + o(x^q).$$

Hence, setting

$$c = \frac{a}{\alpha(q - 2r + 1)},$$

we get

$$(\Phi_* v)(x) = \alpha x^r + o(x^q).$$

Therefore the transformation

$$\Phi(x) = \Phi_{r-2} \circ \Phi_{r-3} \circ \ldots \circ \Phi_1$$

with

$$\Phi_i(x) = x + c_i x^{i+1}, \quad i \in \overline{1, r-2}$$

brings the field $v$ to the form

$$(\Phi_* v)(x) = \alpha x^r + ax^{2r-1} + x^{2r}h(x).$$

By a linear transformation $x \mapsto tx$ we bring it to the form we need. □

Note that the coefficient $d$ in the normal form guaranteed by Theorem 2.1 is invariant: if *two normal forms are conjugate then they coincide.*

Theorem 2.1 gives us a complete list of normal forms of "not too degenerate" vector fields on the line. "Not too degenerate" means that there is $r < \infty$ such that $v^{(r)}(0) \neq 0$. A vector field with

$$v^{(r)}(0) = 0, \quad r = 1, 2, \ldots$$

is called *flat.*

# 3 Formal mappings and vector fields

Let $C^\infty$-vector fields $w$ and $v$ be locally $C^\infty$-conjugate in a neighborhood of the origin,

$$w(x) = (\Phi_* v)(x), \quad \Phi : (U, 0) \to (V, 0).$$

Denote by $\hat{w}(x)$, $\hat{v}(x)$ and $\hat{\Phi}(x)$ the formal Taylor series at zero of the mappings $w, v$ and $\Phi$ respectively. Then

$$\hat{w}(x) = (\hat{\Phi}'(x))^{-1} \hat{v}(\hat{\Phi}(x)).$$

This means that *local $C^\infty$-conjugacy implies the formal conjugacy.*

More exactly, a *formal mapping*

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

is defined to be a formal series

$$F(x) = \sum_{k=0}^{\infty} F^{(k)}(x)$$

where $F^{(k)} : \mathbb{R}^n \to \mathbb{R}^m$ is a homogeneous polynomial mapping of degree $k$. In particular, $F^{(0)}(x) = \mathrm{const} \in \mathbb{R}^n$ is the *free term* of $\Phi$. Usually we consider a formal mappings without free term, i.e. $\Phi^{(0)}(x) = 0$.

In the coordinates

$$x = (x_1 \, \ldots \, x_n) \in \mathbb{R}^n$$

the formal mapping may be written as a series

$$F(x) = \left\{ \sum_{k=0}^{\infty} \sum_{|I|=k} F_I^j x^I \right\}_{j=1}^{m}.$$

Here $I = (I_1, \ldots, I_n)$ is an integer multi-index and

$$|I| = I_1 + \ldots + I_n, \quad x^I = x_1^{I_1} \ldots x_n^{I_n}.$$

The set of all formall mappings will be denoted by $\mathbb{R}[n, m]$. This set is a linear space over $\mathbb{R}$ with respect to the obvious operations.

The set $\mathbb{R}[n, 1]$ of formal series $f : \mathbb{R}^n \to \mathbb{R}^1$ has an obvious structure of an algebra over $\mathbb{R}$. In a usual way define the product $f \cdot F$ where $f \in \mathbb{R}[n, 1]$ and $F \in \mathbb{R}[n, m]$. This means that $\mathbb{R}[n, m]$ has a structure of $\mathbb{R}[n, 1]$- module.

If $F \in \mathbb{R}[n, m]$ is a mapping with $F^{(0)} = 0$ and $\Phi \in \mathbb{R}[m, l]$ is arbitrary then the *composition* $\Phi \circ F \in \mathbb{R}[n, l]$ is defined (*substitution of a series into a series*).

The space $\mathbb{R}_0[n,n]$ with zero free term is a semigroup with respect to composition. The subgroup of invertible mappings consists of the elements

$$\Phi(x) \;=\; Ax \;+\; \sum_{k\geq 2} \Phi^{(k)}(x)$$

with a nonsingular *linear part* $A$, i.e. $\det A \neq 0$. We call such mappings *formal diffeomorphisms*. Given $F \in \mathbb{R}[n,m]$ one can define the partial derivatives

$$\frac{\partial F}{\partial x_i} \;:=\; \sum_k \frac{\partial F^{(k)}(x)}{\partial x_i}$$

and the *Jacobi matrix* (over the ring $\mathbb{R}[n,1]$, i.e. with elements from the ring):

$$F^{'}(x) \;=\; \left(\frac{\partial F^j}{\partial x_i}\right)$$

where

$$F^j(x) \;=\; \sum_{|I|=0}^{\infty} F_I^j x^I, \quad j \;=\; 1,2,\,\ldots\, m$$

are the coordinates of the mapping $F(x) \;=\; \left\{\, F^j(x)\,\right\}_{j=1}^m$.

With a local $C^\infty$-mapping $\Phi \;:\; \mathbb{R}^n \;\to\; \mathbb{R}^m$ we can associate the formal mapping

$$\hat{\Phi} \;:=\; \sum_{k=0}^{\infty} \Phi^{(k)}$$

with

$$\Phi^{(k)}(x) \;=\; \sum_{|I|=k} \frac{1}{I_1!\,\ldots\, I_n!}\frac{\partial^k \Phi(0)}{\partial x^I} x^I.$$

This mapping is called *the Taylor series of* $\Phi$ *at zero.*

**Borel lemma** (see [11]). *For each formal mapping* $F \;\in\; \mathbb{R}[n,m]$ *there is a* $C^\infty$*-mapping* $\Phi \;:\; \mathbb{R}^n \;\to\; \mathbb{R}^m$ *such that* $\hat{\Phi} \;=\; F$.

In other words, the correspondence $\Phi \;\mapsto\; \hat{\Phi}$ is surjective. Obviously, it is not injective: its kernel consists of *flat mappings* $\Phi$, i.e.

$$\frac{\partial^k \Phi(0)}{\partial x^I} \;=\; 0, \quad |I| \;=\; k, \quad k \;=\; 0, 1,\,\ldots\,.$$

A formal mapping $\hat{v} \;\in\; \mathbb{R}_0[n,n]$ may be interpreted as a *formal vector field* giving a *formal system*

$$\frac{dx}{dt} \;=\; \hat{v}(x).$$

The formal transformation $x \mapsto \hat{\Phi}(x)$ transforms it to the system

$$\frac{dx}{dt} = \hat{w}(x)$$

with $\hat{w}(x) = (\hat{\Phi}'(x))^{-1}\hat{v}(\hat{\Phi}(x))$. In this sense the formal vector fields $\hat{v}$ and $\hat{w}$ are called *conjugate.*

Two $C^\infty$-vector fields $v$ and $w$ are called *formally conjugate* if the corresponding formal Taylor series $\hat{v}$ and $\hat{w}$ are conjugate as formal vector fields. Taking into account Borel lemma, one can say that *$C^\infty$-vector fields $v$ and $w$ are formally conjugate if there exists a local $C^\infty$- transformation $\Phi$ such that the difference $\tau = w - \Phi_* v$ is flat at zero.*

Thus, given a local $C^\infty$-vector field $v$, the problem of its normal form contains two parts: an "algebraic" part, which consists in looking for a normal form of the formal v.f. $\hat{v}$ and a "functional" one consisting in the possibility of "killing" a flat residual $\tau$.

# 4 Linearization

The simplest vector field singular at zero is linear $w(x) = Ax$. Its trajectories have the form

$$x(t) = \exp(tA)x$$

and may be easily investigated. This is a reason to start with a possibility of linearizing a vector field.

Following the aforesaid, consider first the corresponding formal problem.

Let $\hat{v}(x) = Ax + \sum_{k\geq 2} v^{(k)}(x)$ be a formal vector field with the linear part $v^{(1)}(x) = Ax$. Let us look for a formal transformation

$$\hat{\Phi}(x) = x + \sum_{k\geq 2} \Phi^{(k)}(x)$$

carrying $\hat{v}$ to the linear $w(x) = Ax$, i.e. such that

$$(\hat{\Phi}'(x))^{-1} A\hat{\Phi}(x) = \hat{v}(x),$$

or $A\hat{\Phi}(x) = \hat{\Phi}'(x)\hat{v}(x)$.

The homogeneous summands of degree $k$ at the left- and right-hand sides have respectively the form

$$A\Phi^{(k)}(x) = (\Phi^{(k)}(x))' Ax + v^{(k)}(x) + \sum_{s=2}^{k-1} (\Phi^{(s)}(x))' v^{(k-s+1)}(x).$$

Hence, if for every $k \geq 2$ one can find $\Phi^{(k)}(x)$ such that

$$A\Phi^{(k)}(x) - (\Phi^{(k)}(x))' Ax = v^{(k)}(x) + \sum_{s=2}^{k-1} (\Phi^{(s)}(x))' v^{(k-s+1)}(x), \qquad (4.1)$$

then $\hat{v}(x)$ is conjugate with $Ax$. The system (4.1) may be solved if the linear operator

$$(L_A \Phi^{(k)})(x) := A\Phi^{(k)}(x) - (\Phi^{(k)}(x))' Ax$$

acting in the space of homogeneous mappings $\Phi^{(k)} : \mathbb{R}^n \to \mathbb{R}^n$ of the degree $k$, is invertible. The operator $L_A$ coincides with the *cohomology operator* (cf. [A2]). To clear up invertibility conditions of $L_A$ let us find its eigenvalues.

Let $\lambda_1 \ldots \lambda_n$ be eigenvalues of $A$ and $z_1 \ldots z_n$ be corresponding eigenvectors. Let the linear functionals $f_1(x), \ldots, f_n(x)$ be dual eigenvectors, i.e.

$$f_i(Ax) = \lambda_i f_i(x), \quad i \in \overline{1,n}.$$

It is clear that

$$f_i(e^{tA}x) = e^{t\lambda_i} f_i(x) \quad (t \in \mathbb{R}).$$

Given a multi-index $I = (I_1, \ldots, I_n)$, $|I| = k$, set

$$\Phi(x) = f_1^{I_1}(x) \ldots f_n^{I_n}(x) z_i.$$

Then $\Phi(x)$ is a homogeneous mapping, $\deg \Phi = |I| = k$, and

$$\Phi(e^{tA}x) = e^{t(\lambda,I)}\Phi(x), \quad A\Phi(x) = \lambda_i \Phi(x),$$

where $(\lambda, I) = \sum_{j=1}^{n} \lambda_j I_j$. Consequently,

$$\frac{d\Phi(e^{tA}x)}{dt}\Big|_{t=0} = \Phi'(x)Ax = (\lambda, I)\Phi(x) \ ,$$

and

$$(L_A\Phi)(x) = (\lambda_i - (\lambda, I))\Phi(x).$$

Hence, the numbers

$$\mu_{I,i} := \lambda_i - (\lambda, I)$$

are eigenvalues of the operator $L_A$. Dimensional arguments show that *these are all the eigenvalues of the operator*. Therefore if all of them are different from zero, then the operator $L_A$ is invertible for every $k \geq 2$, and one can solve the system (4.1) with respect to $\Phi^{(k)}$, $k = 2, 3, \ldots$ by induction as a recursion system. So, we get

**Theorem 4.1.** (Poincaré) *Let*

$$\lambda_i \neq \sum_j \lambda_j I_j, \quad |I| = k, \quad k = 2, 3, \ldots .$$

*Then the vector field* $v(x) = Ax + \ldots$ *is formally conjugate with its linear part* $w(x) = Ax$.

The equality

$$\lambda_i = \sum_j \lambda_j I_j, \quad |I| = k$$

for the eigenvalues of the operator $A$ is called a *resonance* of order $k$.

A possibility of $C^\infty$-linearization under the same condition was first proved by S.Sternberg: *any resonance free* $C^\infty$*-vector field is locally* $C^\infty$*-conjugate with its linear part.*

In fact, a stronger result holds. Namely, let us recall that a vector field $v(x) = Ax + o(x)$ is called *hyperbolic* at the singular point $x_0 = 0$ if $A$ has no pure imaginary eigenvalues, i.e. $\mathrm{Re}\,\lambda_i \neq 0$.

For instance, every resonance free vector field $v(x) = Ax + o(x)$ is hyperbolic.

**Theorem 4.2.** (Sternberg, Chen). *If two hyperbolic at zero* $C^\infty$*-vector fields are formally conjugate at the singular point then they are locally* $C^\infty$*-conjugate.*

Obviously, hyperbolicity is essential. Here is a trivial example. Zero vector field $v(x) = 0$ is formally conjugate with a flat one $\tau(x)$. Any local diffeomorphism $\Phi$ may be taken as a conjugating transformation. On the other hand, zero vector field is not smoothly conjugate with a vector field $\tau \neq 0$.

# 5 Resonant normal form

In general, linearization is impossible in the presence of resonance, but it is possible to determine a certain normal form.

Given a formal vector field

$$\hat{v}(x) = Ax + \sum_{k \geq 2} v^{(k)}(x)$$

let us look for a "simple" form for the formal vector field $\hat{w} = \hat{\Phi}_* \hat{v}$ with a formal transformation

$$\hat{\Phi}(x) = x + \sum_{k \geq 2}^{\infty} \Phi^{(k)}(x).$$

Writing $\hat{w}$ in the form

$$\hat{w}(x) = Ax + \sum_{k \geq 2} w^{(k)}(x)$$

we obtain for its components the recursion system

$$w^{(k)}(x) \;=\; L_A\Phi^{(k)}(x) \;+\; v^{(k)}(x) \;+\; R_k, \quad k \;=\; 2,3,\,\ldots \tag{5.1}$$

where $R_k$ is an expression depending on summands of degree $\leq\, k-1$ :

$$R_k \;=\; R_k(v^{(2)},\, \ldots\, v^{(k-1)}, w^{(2)}\, \ldots\, w^{(k-1)}, \Phi^{(2)}\, \ldots\, \Phi^{(k-1)}).$$

In particular, $R_2 \;=\; 0,$ hence

$$w^{(2)}(x) \;=\; L_A\Phi^{(2)}(x) \;+\; v^{(2)}.$$

The operator $L_A$ is as above, but in the presence of resonance it is not invertible and one can not set $w^{(k)} \;=\; 0.$

To construct a normal form let us fix in $\mathbb{R}^n$ the standard basis $e_1\, \ldots\, e_n.$ Let $A \;=\; (a_{ij})$ be the matrix of the linear part in the basis.

**Theorem 5.1.** *Given a formal v.f.* $\hat{v}(x) = Ax + \sum\limits_{k\geq 2} v^{(k)}(x),$ *there is a formal transformation* $\hat{\Phi}(x) \;=\; x + \ldots$ *bringing* $\hat{v}$ *to the form* $(\hat{\Phi}_* v)(x) \;=\; Ax \,+\, h(x)$ *where* $h$ *is a formal v.f. with zero linear part commuting with* $A^T,$ *i.e.*

$$A^T h(x) \;=\; h^{'}(x) A^T x, \tag{5.2}$$

*where* $A^T$ *is the transposed matrix.*

A proof of Theorem 5.1 is based on the following lemma.

Let $L\, :\, \mathcal{E}_1 \;\to\; \mathcal{E}_2$ be a linear operator between Euclidean spaces $\mathcal{E}_1$ and $\mathcal{E}_2.$ Set

$$u.z \;=\; z \;+\; Lu \quad (u \;\in\; \mathcal{E}_1,\; z \;\in\; \mathcal{E}_2). \tag{5.3}$$

By this formula the additive group of the space $\mathcal{E}_1$ acts in $\mathcal{E}_2.$ There is a simple normal form for this action:

**Lemma 5.1.** *Given an element* $z \;\in\; \mathcal{E}_2$ *there exists a unique element* $w \in \mathcal{E}_2$ *equivalent to* $z$ *in the sense* (5.3) *and satisfying the equation* $L^* w \,=\, 0,$ *where* $L^*$ *is the dual operator.*

**Proof.** The direct decomposition is well known

$$\mathcal{E}_2 \;=\; \mathrm{Im}\, L \;+\; \mathrm{Ker}\, L^*.$$

Given $z \;\in\; \mathcal{E}_2,$ let us decompose it into the sum

$$z \;=\; v \;+\; w, \quad v \;\in\; \mathrm{Im}\, L,\; w \;\in\; \mathrm{Ker}\, L^*.$$

There exists an element $u \;\in\; \mathcal{E}_1$ such that $v \;=\; -Lu.$ Then

$$u.z \;=\; v + w \;-\; Lu \;=\; w. \qquad \square$$

**Proof of Theorem 5.1.** Let us denote by $\mathcal{E}^{(k)}$ the space of all polynomial homogeneous mappings

$$\Psi(x) \;=\; \left\{ \sum_{|I|=k} \Psi_I^j x^I \right\}_{j=1}^{n}, \qquad \Psi_I^j \;\in\; \mathbb{R}$$

of degree $k$. Introduce in $\mathcal{E}^{(k)}$ an inner product by the formula

$$(\Phi, \Psi) \;=\; \sum_{j,I} I! \Phi_I^j \cdot \Psi_I^j, \quad I! \;=\; I_1! \,\ldots\, I_n!. \tag{5.4}$$

Then $\mathcal{E}^{(k)}$ becomes an Euclidean space. Following Lemma 5.1, choose $\Phi^{(2)} \in \mathcal{E}^{(2)}$ such that $w^{(2)} \in \mathrm{Ker} L_A^*$. Then, fixing $\Phi^{(2)}$ and $w^{(2)}$, choose $\Phi^{(3)}$ such that

$$w^{(3)}(x) \;=\; L_A \Phi^{(3)}(x) \;+\; v^{(3)}(x) \;+\; R_3(v^{(2)}, \Phi^{(2)}, w^{(2)})(x)$$

belongs to $\mathrm{Ker}\, L_A^*$. Continuing by induction, we build a transformation $\hat{\Phi}(x) = x \;+\; \sum\limits_{k \geq 2}^{\infty} \Phi^{(k)}(x)$ bringing $\hat{v}$ to a v.f. $\hat{w}$ such that

$$L_A^* w^{(k)}(x) \;=\; 0, \quad k \;=\; 2, 3, \;\ldots\;.$$

To complete the proof of Theorem 5.1 it remains to check that the dual operator $L_A^*$ has the form

$$L_A^* w^{(k)}(x) \;=\; A^T w^{(k)}(x) \;-\; (w^{(k)}(x))' A^T x.$$

It is sufficient to check that

$$(Aw(x), \psi(x)) \;=\; (w(x), A^T \psi(x)) \tag{5.5}$$

and

$$((w(x)' Ax, \psi(x)) \;=\; (w(x), (\psi(x))' A^T) \tag{5.6}$$

In turn, it is sufficient to check (5.5) and (5.6) for the basis elements

$$w(x) \;=\; x^I e_p, \quad \psi(x) \;=\; x^J e_q.$$

This is very simple for (5.5), since

$$(Aw, \psi) \;=\; 0 \quad (I \;\neq\; J), \quad (Aw, \psi) \;=\; I! a_{qp} \;=\; (w, A^T \psi)$$

for $I \;=\; J$.

Further, for (5.6) we have

$$(w^{'}(x)Ax, \psi(x)) \ = \ \begin{cases} 0, & J \neq I - e_k + e_j \text{ or } p \neq q \\ I! \cdot \sum\limits_{k=1}^{n} a_{kk} I_k, & J = I, \ p = q \\ J! \cdot a_{kj} I_k, & J = I - e_k + e_j, \ p = q \end{cases}$$

The same result gives us the expression $(w(x), \psi^{'}(x)A^T x)$. □

It is easy to check that the equality $Bh(x) = h^{'}(x)Bx$ is equivalent to the equality

$$h(e^{tB}x) \ = \ e^{tB}h(x) \quad (t \in \mathbb{R}). \tag{5.7}$$

Indeed, set

$$\gamma(t) \ = \ e^{-tB}h(e^{tB}x).$$

Then $\gamma(0) = h(x)$. Further,

$$\gamma^{'}(t) \ = \ e^{-tB}h^{'}(e^{tB}x)e^{tB}Bx \ - \ e^{-tB}Bh(e^{tB}x) \ =$$

$$= \ -e^{-tB}\left[ Bh(e^{tB}x) \ - \ h^{'}(e^{tB}x)Be^{tB}x \right] \ = \ 0$$

Hence, $\gamma(t) = \text{const} = h(x)$. Conversely, differentiating (5.7) and setting $t = 0$, we get $Bh(x) = h^{'}(x)Bx$.

In particular, let $A^T$ be triangular:

$$A^T \ = \ \begin{pmatrix} \lambda_1 & & & * \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_n \end{pmatrix}.$$

Then, setting $B = A^T$ in (5.7), we obtain

$$\sum_I h_I^i (e^{t\lambda_1}x_1 \ + \ \ldots)^{I_1} \ \ldots \ (e^{t\lambda_n}x_n)^{I_n} \ = \ e^{t\lambda_i} \sum_I h_I^i x^I, \quad i \in \overline{1,n}.$$

Equating the corresponding coefficients, we get

$$e^{t(I,\lambda)}h_I^i \ = \ e^{t\lambda_i}h_I^i.$$

Therefore, (5.2) implies the condition

$$\lambda_i \neq \sum_{k=1}^{n} I_k \lambda_k \ \Rightarrow \ h_I^i \ = \ 0.$$

This condition, in turn, is equivalent to the equality

$$h(e^{tD}x) = e^{tD}h(x), \quad D = \operatorname{diag}(\lambda_1 \ldots \lambda_n) \tag{5.8}$$

or

$$h'(x)Dx = Dh(x).$$

A vector field satisfying this equality is called *resonant normal form* or *Poincaré-Dulac normal form.*

**Example 5.1.** Consider the normal form (5.2) for the two-dimensional case. There are the following possibilities for eigenvalues $\lambda_1, \lambda_2 \in \operatorname{spec} A$.

a) $\lambda_1 \neq \lambda_2$ are real. Then one can account that $A = \operatorname{diag}(\lambda_1, \lambda_2)$ and the normal form (5.2) is resonant:

$$\dot{x}_1 = \lambda_1 x_1 + \sum_{p+q\geq 2} h^1_{pq} x_1^p x_2^q$$

$$\dot{x}_2 = \lambda_2 x_2 + \sum_{p+q\geq 2} h^2_{pq} x_1^p x_2^q$$

where $h^j_{pq} = 0$ as soon as $\lambda_j \neq p\lambda_1 + q\lambda_2$.

b) $\lambda_1 = \lambda_2 = \lambda \neq 0$. This case is resonance free and the normal form is linear

$$\dot{x} = Ax.$$

c) $\lambda_1 = \lambda_2 = 0$. In this case either $A = 0$ and (5.2) gives us nothing or $A$ is similar to the Jordan block

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then the resonant normal form is empty, since $D = 0$ and (5.8) holds for any $h$. But (5.2) with $A = N$ implies additional conditions:

$$\dot{x}_1 = x_2 + x_1\psi(x_1)$$

$$\dot{x}_2 = x_2\psi(x_1)$$

where $\psi(z)$ is a formal series of a single variable.

d) $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \overline{\lambda_1}$ with $\beta \neq 0$. If $\alpha \neq 0$ then the resonance is absent, and the normal form is linear. In the case $\alpha = 0$ the following resonances arise:

$$\lambda_1 = (p + 1)\lambda_1 + p\lambda_2, \quad \lambda_2 = p\lambda_1 + (p + 1)\lambda_2.$$

The matrix $A$ is similar to the anti-symmetric one

$$S = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Then the matrix

$$e^{tS} = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}, \quad t \in \mathbb{R}$$

is orthogonal. In the complex coordinates

$$z = x_1 + ix_2, \quad \overline{z} = x_1 - ix_2$$

we have

$$e^{tS}x = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{i\beta t}z.$$

Hence, setting

$$h(x) = \tilde{h}(z,\overline{z}) = \sum \tilde{h}_{pq} z^p \overline{z}^q, \quad \tilde{h}_{pq} \in \mathbb{C}$$

and substituting $B = S$ in (5.7), we obtain

$$\tilde{h}(e^{i\beta t}z, e^{-i\beta t}\overline{z}) = e^{i\beta t}\tilde{h}(z,\overline{z}),$$

or

$$\tilde{h}(z,\overline{z}) = \sum a_p z^{p+1}\overline{z}^p = zg(|z|^2).$$

So, (5.2) in this case gives us

$$\dot{z} = z(i\beta + g(|z|^2))$$

with a complex formal series $g(u) = \sum\limits_{k\geq 1} g_k u^k$, $g_k \in \mathbb{C}$ . In the case $\mathrm{Re}\, g(u) \neq 0$ the point $x = 0$ is called *focus*. The case $\mathrm{Re}\, g(u) = 0$ is degenerate. The trajectories satisfy the equation

$$\frac{d}{dt}|z|^2 = 0,$$

i.e., they are closed: $|z| = \mathrm{const}$. The point $x = 0$ is called *center* for the v.f. of such a type.

The condition (5.2) may be introduced for $C^\infty$-mappings, not only for formal ones. So, the question arises, *whether every* $C^\infty$-v.f.

$$v(x) = Ax + \ldots$$

*may be reduced to the form*

$$w(x) = Ax + h(x), \quad h'(x)A^T x = A^T h(x)$$

*by a* $C^\infty$*-local transformation* $\Phi$ ?

By a slight modification of the proof of the Borel lemma one can get the following generalization:

*Given a formal mapping* $\hat{h}(x)$ *satisfying* (5.2), *there exists a local* $C^\infty$-*mapping* $h(x)$ *satisfying* (5.2) *as well, whose formal Taylor series at zero is equal to* $\hat{h}(x)$.

Combining the latter assertion with Theorem 5.1 we obtain

**Corollary 5.1.** *Given a local* $C^\infty$-*v.f.* $v(x) = Ax + \ldots$, *there exists a local* $C^\infty$-*transformation* $\Phi(x)$ *bringing* $v$ *to the form* $(\Phi_* v)(x) = Ax + h(x) + \tau(x)$, *where* $h(x) = o(x)$ *satisfies* (5.2) *and* $\tau$ *is a flat residual.*

If $v$ is hyperbolic at zero then $\tau(x)$ may be "killed" by an additional $C^\infty$-transformation (Sternberg-Chen Theorem). Below (Corollary 7.1) we will see that the same holds in some important non-hyperbolic cases.

# 6 Normal form equation. Main Theorem

It is clear that the normal form (5.2) for formal v.f. is not "canonical". This means that two formal v.f. having different normal forms may be conjugate. For instance, if $A = 0$ then every formal v.f. satisfies (5.2). The natural aim is to find more exact normal forms.

To this end we introduce the following notations.

Given a polynomial $P(x) = \sum P_I x^I$ we denote $P(\frac{\partial}{\partial x})$ the differential operator in the space of formal series $U(x) = \sum\limits_J U_J x^J$

$$P(\frac{\partial}{\partial x})U := \sum_I P_I \frac{\partial^I U}{\partial x_1^{I_1} \ldots \partial x_n^{I_n}}.$$

By the same way one can define the operator $P(\frac{\partial}{\partial x})$ for a matrix $P = (P_{ij})$ over the ring of polynomials. Recall that by $B^T$ we denote the matrix transposed to $B = (b_{ij}^j)$.

Further, given a polynomial v.f. $P(x) = (P^1(x) \ldots P^n(x))$ we denote by $\mathcal{L}P$ the operator, acting in the space of all formal v.f. $w = (w^1 \ldots w^n)$ by the formula

$$(\mathcal{L}P)w = \left\{ \sum_{i=1}^n x_i P^i(\frac{\partial}{\partial x}) w^j(x) \right\}_{j=1}^n - \left( P^{'}(\frac{\partial}{\partial x}) \right)^T w(x).$$

The operator $\mathcal{L}P$ is dual in the sense of the inner product (5.4) to the operator

$$(HP)(w) = \{P, w\} = P^{'} w - w^{'} P.$$

**Example 6.1.** Let $n = 1$. Then

$$(\mathcal{L}P)w = \sum_k \left( P_k \frac{\partial^k w(x)}{\partial x^k} x - k P_k \frac{\partial^{k-1} w(x)}{\partial x^{k-1}} \right)$$

for the polynomial $P(x) = \sum P_k x^k$ and a formal mapping $w$.

In the linear case $P(x) = Ax$ the operator $(\mathcal{L}P)(w)$ concides with $(L_A^* w)(x)$ from Sections 4.5.

Given a formal v.f. $w = \sum_{j=1}^{\infty} w^{(j)}(x)$ we denote by $\mathrm{pr}_k w$ its projection on the space of polynomial mappings of degree $\leq k$, i.e.

$$\mathrm{pr}_k w = \sum_{j \leq k} w^{(j)}(x).$$

**Definition 6.1.** *A formal vector field* $w = \sum_{j \geq 1} w^{(j)}(x)$ *is called a normal form if its summands satisfy the equations*

$$(\mathcal{L}\mathrm{pr}_{k-1} w) w^{(k)} = (\mathcal{L}\mathrm{pr}_{k-1} w)(v_{k-1}), \quad k = 2, 3, \ldots \tag{6.1}$$

*where* $v_{k-1}$ *are some polynomial v.f. of degree* $\leq k-1$.

We call (6.1) the *normal form equation.*

In particular, if $w$ satisfies the equalities

$$(\mathcal{L}\mathrm{pr}_{k-1} w) w^{(k)} = 0, \quad k = 2, 3, \ldots \tag{6.2}$$

then one can set $v_{k-1} = 0$ in (6.1), and hence $w$ is a normal form.

**Example 6.2.** Let $w(x) = Ax + w^{(p)}(x)$ with a fixed $p \geq 2$. If $w^{(p)}$ satisfies (5.2), then (6.2) holds and $w$ is a normal form.

But, generally speaking, a normal form does not satisfy (6.2).

Since

$$\mathcal{L}\mathrm{pr}_{k-1} w = \sum_{j \leq k-1} \mathcal{L} w^{(j)},$$

one can rewrite (6.1) in the form

$$(\mathcal{L} w^{(j)})(w^{(k)}) = \sum_{s=1}^{k-1} (\mathcal{L} w^{(s)})(v^{(k-j+s)}), \quad k = 2, 3, \ldots \tag{6.3}$$

for $j = 1, \ldots k-1$ and $j - s \geq 1$.

In particular, let $w^{(j)} = 0$, $j \leq p-1$. Set $j = p$ in (6.3). Since $s \geq p$ and at the same time $p - s \geq 1$, we obtain $(\mathcal{L} w^{(p)})(w^{(k)}) = 0$, $k = 2, 3, \ldots$. Hence, *if a v.f.* $w = w^{(p)} + \ldots$ *is a normal form then*

$$(\mathcal{L} w^{(p)}) w = 0. \tag{6.4}$$

**Example 6.3.** If a v.f. $w(x) = \pm x^p + \sum\limits_{k \geq p+1} h_k x^k$ on the line is a normal form then (6.4) gives us

$$h_k \frac{\partial^p x^k}{\partial x^p} x - p h_k \frac{\partial^{p-1} x^k}{\partial x^{p-1}} = 0, \quad k = p+1, \dots ,$$

or

$$h_k(k(k-1)\dots(k-p+1) - pk(k-1)\dots(k-p+2)) = 0.$$

Hence, $h_k = 0 \ (k \neq 2p-1)$ (cf. Theorem 2.1).

The main formal result consists in the possibility of reducing any formal v.f. to the normal form.

More exactly, denote by $\mathbf{G}$ the group of all formal transformations $\Phi(x) = x + \dots$ i.e. $\Phi'(0)$ is the identity matrix.

**Theorem 6.1.** *Given a formal v.f.* $\hat{v}$, *there exists a unique normal form* $\hat{w}$ *which is conjugate with* $\hat{v}$ *with respect to the group* $\mathbf{G}$, *i.e.*

$$\hat{w} = \hat{\Phi}_* \hat{v}, \quad \hat{\Phi} \in \mathbf{G}.$$

Thus, in contrast to the "noncanonical" normal form (5.2), the normal form (6.1) is "canonical". Because of uniqueness its coefficients form a complete system of invariants with respect to the action of the group $\mathbf{G}$.

Taking into account the property (6.4) of a normal form, we get

**Corollary 6.1.** *A formal v.f.* $\hat{v} = v^{(p)}(x) + \dots$ *by a formal transformation* $\hat{\Phi}(x) = x + \dots$ *may be reduced to a v.f.* $\hat{w} = v^{(p)} + \sum\limits_{k \geq p+1} w^{(k)}$ *satisfying the equality*

$$(\mathcal{L} v^{(p)}) \hat{w} = 0.$$

The latter is a generalization of Theorem 5.1 which corresponds to the case $p = 1$.

The proof of Theorem 6.1 consists in sequential applications of Lemma 5.1 for Euclidean spaces $\mathcal{E}_k$ of polynomial v.f. of degree $\leq k$, $k = 2, 3, \dots$ endowed with the inner product (5.4). It should be added to the calculations of Theorem 5.1 that the operator in $\mathcal{E}_k$ dual to multiplication on a variable, say $x_1$, is the derivation $\frac{\partial}{\partial x_1}$. A complete proof is given in [B4].

Usually a calculation of normal forms is rather complicate. Therefore not many examples are known up to now. We present some of them for two-dimensional v.f. $v(x) = Ax + \dots$, $x \in \mathbb{R}^2$.

**Example 6.4** ([9]). Let eigenvalues of $A$ be $\lambda_1 = 0$, $\lambda_2 = 1$. Then $\hat{v}$ may be formally reduced to the system

$$\dot{x}_1 = x_1^{m+1}\left(\sum_{j=0}^{m-l} a_j x_1^j + a_m x_1^m\right)$$

$$\dot{x}_2 = x_2(1 + b x_1^l),$$

which is a normal form for every real $a_j, a_m, b$ and integer $l \le m < \infty$. All these parameters are invariant and depend on $v$ only. The normal form is polynomial.

**Example 6.5** ([17]). Let $\lambda_1 = -\lambda_2 = i\beta$ as in Example 5.1, d). In the non-degenerate case $\operatorname{Re} g \neq 0$ the formal normal form is polynomial as well:

$$\dot{z} = z(i\beta \pm i|z|^{2l} + b_0|z|^{2p} + \ldots + b_p|z|^{4p})$$

with some integer $l \le p < \infty$ and real $b_0 \ldots b_p$, $b_0 \neq 0$.

# 7 $\omega$-definiteness of local v.f.

It follows from Theorem 6.1 and Borel lemma that every local $C^\infty$-v.f. by a local $C^\infty$-transformation $\Phi$ may be reduced to a v.f. $w = \Phi_* v$ such that the Taylor series $\hat{w}$ at zero is a normal form.

Moreover, by Sternberg-Chen Theorem, if $v_1$ and $v_2$ are two hyperbolic v.f. with the same normal form of the Taylor series then they are $C^\infty$-conjugate. In other words, *the formal normal form gives us a complete classification of local hyperbolic v.f.* with respect to $C^\infty$-transformations $\Phi(x) = x + \ldots$ .

In the non-hyperbolic case this is not true: it may happen that vector fields $v_1$ and $v_2$, having the same formal normal form of Taylor series are not $C^\infty$-conjugate. Therefore, it is important to clear up in what cases, aside from hyperbolic, formal conjugacy implies $C^\infty$-conjugacy.

In the general case a local v.f. $v$ is called *$\omega$-definite* if any v.f., which is formally conjugate with $v$, is $C^\infty$-conjugate with $v$. For example, a hyperbolic v.f. is $\omega$-definite (Sternberg-Chen Theorem).

If $v(x) = Ax + \ldots$ is nonhyperbolic, then there exists a subspace $L_c \subset \mathbb{R}^n$, invariant with respect to $A$, such that all eigenvalues of the restriction $A|L_c$ lie on the imaginary axis. In our context without loss of generality one can assume $L_c$ to be invariant with respect to v.f. $v$. In fact there is a local $C^\infty$-transformation $\Phi$ and a flat residual such that $L_c$ is invariant with respect to the v.f. $w = \Phi_* v + \tau$. In this sense the local manifold $\Phi^{-1}(L_c)$ is *formally invariant center manifold* for the v.f. $v$.

Further investigation of $\omega$-definiteness is based on the following "Sternberg-Chen Theorem modulo center manifold":

**Theorem 7.1.** *Suppose that two local v.f. $v_1$ and $v_2$ have a common center manifold $M$ and the difference $v_1 - v_2$ is flat on $M$. Then $v_1$ and $v_2$ are locally $C^\infty$-conjugate.*

Thus, the local conjugacy of $v_1$ and $v_2$ depends on their structure on the center manifold. It turns out, that, if $\dim L_c \leq 2$ and the restriction of $v$ to $L_c$ is not "too degenerate" then $v$ is $\omega$-definite.

More exactly, let us define an "exceptional subset" $\mathbf{N}$ of v.f. with a central manifold $L_c$ of dimensional $\leq 2$.

If $\dim L_c = 1$ then $v \in \mathbf{N}$ iff $\hat{v}\,|L_c \neq 0$. Hence, in this case $v_c := \hat{v}\,|L_c$ is conjugate with the normal form $w(x) = \pm x^p + \alpha x^{2p-1}$, $p \geq 2$ (cf. Theorem 2.1).

In the case $\dim L_c = 2$ we suggest that $v \in \mathbf{N}$ iff the eigenvalues of the restriction $A\,|L_c$ are different from zero and, hence are $\pm i\beta$, $\beta \neq 0$. Besides, we require from the restriction $v_c$ not to be a "formal center", i.e. $\mathrm{Re}\, g \neq 0$ as in item d) of Example 5.1.

**Theorem 7.2.** *Let $v \notin \mathbf{N}$. Then $v$ is $\omega$-definite.*

Therefore, coefficients of the formal normal form of v.f. $v \notin \mathbf{N}$ form a complete system of invariants with respect to $C^\infty$-classification. Besides, it follows from Corollary 5.1 and Theorem 7.2:

**Corollary 7.1.** *A local $C^\infty$-vector field $v(x) = Ax + \ldots \notin \mathbf{N}$ by a local $C^\infty$-transformation $\Phi$ may be reduced to the form*

$$w(x) = Ax + h(x), \quad h'(x)A^T x = A^T h(x).$$

The v.f. from Example 6.4 satisfies conditions of Theorem 7.2, if not all of the parameters $a_j$, $a_m$ are equal to zero. Hence, $v$ is $C^\infty$-conjugate with the polynomial normal form. Similarly, the v.f. from Example 6.6 also may be reduced to the polynomial normal form by a local $C^\infty$-transformation.

# 8 Comments.

The concept and first results on normal forms for v.f. started from Poincaré [13]. Theorem 2.1 first arises in [16]. The "noncomplete" normal form (5.2) was first stated in [4]. The normal form equation was found in [5], [6]. Theorem 4.2 in the non-resonant case was proved by Sternberg [14] and in the general hyperbolic case by Chen [10]. Theorem 7.1 was first stated for maps in [3] and then Samovol [15] proved it for v.f. (cf. [12]). Theorem 7.2 was stated in [7].

Our methods, both formal and local, are applicable for other classification problems, such as symmetric and reversible systems, hamiltonian systems, and so on (see, for instance, [8])

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