The reduction of equivariant dynamics to the orbit space for compact group actions

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Abstract

Symmetry introduces degeneracies in dynamical systems, as well as in bifurcation problems. An "obvious" idea in order to remove these degeneracies is to project the dynamics onto the quotient space obtained by identifying points in phase space which lie in the same group orbits (the so-called *orbit space*). Unfortunately, several difficulties raise when one tries to implement this idea. First, the orbit space is not, in general a manifold. Second, how does one explicitely realize the orbit space, and how does one compute and analyze the projected dynamics? In this paper I will describe the methods which have been developped in order to answer these questions, and I will show on three examples how they apply. We shall see that, although not always suitable to treat equivariant dynamics, these methods sometimes lead to insightful reductions.

1 Introduction

We consider a vector field X defined in the euclidean space $\mathbb{R}^n$ on which acts a Lie subgroup G of $O(n, \mathbb{R})$, the Lie group of real orthogonal $n \times n$ matrices. We shall assume that X is G-*equivariant*:

$$X(gx) = gX(x) \text{ for all } x \in \mathbb{R}^n \text{ and } g \in G. \tag{1}$$

We could replace $\mathbb{R}^n$ above with a finite dimensional manifold M, assuming now that G is a compact group acting on M via a smooth action $\phi : G \times M \to M$. Equivariance now means that $\phi^* X = X$. However the following fact is known (Theorem of Mostow-Palais, see Chossat and Lauterbach [7]): any smooth action of a compact Lie group on a manifold M can be embedded into a smooth linear orthogonal action on a vector space. It is therefore no loss of generality to consider an equivariant dynamical system in $\mathbb{R}^n$.

Instead of X, we could also consider a smooth diffeomorphism Φ in $\mathbb{R}^n$, satisfying the G-equivariance relations.

$$\Phi(gx) = g\Phi(x) \text{ for all } x \in \mathbb{R}^n \text{ and } g \in G.$$

Most of the techniques exposed in this paper will apply to Φ, but we let it to the reader to check this assertion.

Throughout this paper, we will set V for $\mathbb{R}^n$. Given an element $x \in V$, the G-*orbit* of x is the equivalence class

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

The *type* of an orbit is the conjugacy class of the stabilizers (or isotropy subgroups) of its elements. Recall that the stabilizer of an element $x \in V$ is the Lie subgroup

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

If $x' = h \cdot x$ for some $h \in G$, then $G_{x'} = hG_xh^{-1}$. Each G-orbit is therefore characterized by its type. Moreover, each orbit is a smooth (compact) manifold, whose dimension equals $\dim G - \dim H$ where H is an element of its orbit type.

The union of orbits of same type is itself a smooth manifold which we call a *stratum*, and there is a 1-1 correspondence between strata and orbit types. The space V is therefore the disjoint union of strata. However the strata don't all have the same dimension in general. For this reason the *orbit space* V/G, i.e. the quotient of the space by the equivalence relation: $x \sim y$ iff x and y are in the same G-orbit, is not a manifold in general. It is however a manifold when the action of the group is free, that is when there is only one orbit type (and hence only one stratum).

That the orbit space is not a manifold can easily be seen on the simplest possible non-trivial example: the group $\mathbb{Z}_2$ acting on $\mathbb{R}$ by reflection: $x \mapsto -x$. There are two strata: $\{0\}$ (orbit type $\mathbb{Z}_2$) and $\mathbb{R} \setminus \{0\}$ (orbit type: the trivial group I). The orbit space is therefore identical to the half-line $[0, +\infty)$ (the orbit of each $x \neq 0$ can be identified with $|x|$ and the orbit of 0 is a singleton). This example can be generalized to any dimension as follows: let $G = SO(n)$ act naturally in V. There are two strata: $\{0\}$ (orbit type $SO(n)$) and $V \setminus \{0\}$ (orbit type $SO(n-1)$). It is easy to observe that to each isotropy subgroup isomorphic to $SO(n-1)$ is associated one and only one eigendirection in V. In other words the set of elements which have isotropy subgroup G_x ($x \neq 0$) is the line generated by x minus $\{0\}$ (however the isotropy subgroup of the origin is $SO(n)$, which contains G_x). The orbit space is therefore identical to $[0, +\infty)$ (same argument as above).
Now let X be an $SO(n)$-equivariant vector field. Let us write $x = ru$ in polar coordinates, i.e. $r \geq 0$ and $u \in S^{n-1}$. For any $g \in G_x$, we have

$$g \cdot X(ru) = X(g \cdot ru) = X(ru),$$

so that $X(ru)$ has same isotropy subgroup as x (or equivalently, as u). By the remarks above, it follows that we can write

$$X(ru) = R(r)u$$

where $R : \mathbb{R}^+ \to \mathbb{R}^+$ (proof is left to the reader). Now let us write the differential equation $\dot{x} = X(x)$ in the polar coordinates:

$$\dot{r}u + r\dot{u} = R(r)u$$

Thanks to the fact that $\|u\| = 1$, this system reduces to the scalar equation

$$\dot{r} = R(r), \quad r \geq 0,$$

which we can see as the equation for the dynamics *reduced to the orbit space*. In this simple example, we have been able to explicitely build the orbit space and to reduce the dynamics to it. Our aim in the next two sections will be to expose a general method to achieve the same task for general compact group actions, and to list some basic properties of this reduced dynamics. The orbit space reduction can be a highly nontrivial process which in many cases is unnecessary, as it transpires from "classical" equivariant bifurcation theory, see Golubitsky, Stewart and Schaeffer[14]. In certain situations however, it can at least highlight the geometrical features of the problem and sometimes help solving it. In section 4 we shall review some examples for which this method gives such an insight on classical problems of equivariant dynamics and bifurcation.

2 Compact group actions

2.1 General facts about compact group actions

In this section we will expose the basics of compact group actions. Most results are proven elsewhere, see notably, in the framework of equivariant dynamical systems, Chossat and Lauterbach [7] and references therein. Other references which covers part of this material are Sartori [21] in the framework of variational problems, and Cushman and Bates [10] in the framework of mechanical systems. We restrict here to linear actions in V of closed subgroups of $O(n)$, hence avoiding to some extent introducing the machinery of group representation theory.

Definition 2.1 *An* orbit type *is a conjugacy class of isotropy subgroups of the G-action.*
To each orbit type τ is univocally associated a stratum S_τ *of points in V:*

$$S_\tau = \{x \in V \mid G_x \in S_\tau\}.$$

Orbit types can be partially ordered by group inclusion:

$$\tau < \tau' \iff \text{there exist } H \in \tau \text{ and } H' \in \tau' \; s.t. \; H \subset H'.$$

It is custom to denote by $[H]$ (the conjugacy class of H in G) the orbit type with representative the isotropy subgroup H, or even to denote it by H itself when there is no risk of confusion.
For compact groups, the number of orbit types is always finite, and there exists a smallest orbit type, i.e. one which is contained in all the orbit types. We call it τ_0. In many cases τ_0 reduces to one element (the trivial group). The stratum of smallest orbit type is called the *principal stratum*.
Clearly the strata form a *partition* of V.

Theorem 2.2 *Each stratum is a smooth (C^∞) manifold. The principal stratum is an open subset of V. If $\tau < \tau'$, then $S_{\tau'}$ lies in the closure of S_τ.*

It follows that the space V is *stratified* by orbit types. The proof of this theorem is a consequence of the invariant slice theorem of Palais which is briefly introduced below.

Related to the notion of stratum is the notion of isotropy subspace. Given an isotropy subgroup H, we set

$$\mathrm{Fix}(H) = \{x \in V \mid H \cdot x = x\}.$$

This is a vector space, but points in it can have higher isotropy than H. It is easy to see that the largest subgroup of G which acts in $\mathrm{Fix}(H)$ is the normalizer of H, $N(H)$. The quotient group $N(H)/H$ acts *faithfully* in $\mathrm{Fix}(H)$, meaning that the kernel of the action reduces to the trivial element (the identity matrix). However it must be emphasized that the corresponding stratification of $\mathrm{Fix}(H)$ is *not always* identical to the intersection of the stratification of V with $\mathrm{Fix}(H)$.

Example 2.3 Consider the dihedral group D_3 acting in $\mathbb{R}^2$. This is the symmetry group of the equilateral triangle, therefore it is immediate that there are 3 strata:

- $\{O\}$ (the "barycenter" is fixed by all D_3);
- The union of the 3 axes of symmetry minus $\{O\}$ (orbit type = class formed by the three reflection groups through the axes of symmetry of the triangle);
- complement in $\mathbb{R}^2$ of the above two strata (orbit type = trivial group).

We leave it to the reader to find what the isotropy subspaces are and to check the various properties listed above.

Let $G \cdot x$ be the orbit of a point $x \in V$ under the action of G. As we claimed in the introduction, this is a compact manifold, of dimension equal to $\dim G - \dim G_x$. From the above theorem it follows that there exists a neighborhood of $G \cdot x$ such that all orbits in this neighborhood have dimension greater than or equal to $\dim G \cdot x$. When the orbit is a nontrivial manifold ($\dim > 0$) it is important to have a precise description of the structure of a neighborhood of this orbit. For this we first need to understand the structure of the orbit $G \cdot x$ itself.
Let m be the dimension of G as a manifold (Lie group). This means that, in a neighborhood of the identity (the neutral element for a matrix group), G is smoothly parametrized by m real parameters $s_1, \ldots, s_m$: we write $g = g(s_1, \ldots, s_m)$, and $g(0) = Id$. From the assumption the derivatives

$$A_1 = \frac{\partial g}{\partial s_1}(0), \ldots, A_m = \frac{\partial g}{\partial s_m}(0),$$

are linearly independent matrices and generate the so-called *Lie algebra* of G. Here the Lie structure is simply the commutation of matrices, which we write

$$[A, B] = A \cdot B - B \cdot A.$$

It is now clear that the tangent space to $G \cdot x$ at x is generated by the vectors

$$A_1 \cdot x, \ldots, A_m \cdot x.$$

Suppose $\dim G_x = k > 0$. After a suitable choice of coordinates, this means that

$$g(s_1, \ldots, s_k, 0, \ldots, 0) \cdot x = x$$

for all $s_1, \ldots, s_k$ in a neighborhood of 0. Therefore we have that

$$A_1 \cdot x = \cdots = A_k \cdot x = 0,$$

which proves the formula about the dimension of the orbit $G \cdot x$.

Definition 2.4 *We denote the tangent space to $G \cdot x$ at x by T_x and its orthogonal complement by N_x. We call N_x the* normal slice *to the orbit at x.*

Note that N_x is a vector subspace of V, because x is orthogonal to T_x (G is a group of isometries). For the same reasons, the isotropy subgroup G_x acts in N_x.
Let U be a G_x-invariant neighborhood of x in N_x. There is a natural projection of the set $G \times U$ onto a neighborhood $\mathcal{U}$ of $G \cdot x$ in V. This map is not a bijection however, unless G_x acts trivially in N_x. There is nevertheless a natural identification of the *tubular neighborhood* $\mathcal{U}$ with the so-called G_x-twisted product of G by U, as follows. We simplify the notation by setting $H = G_x$.

Definition 2.5 *We call* twisted product *of G by U, and we write $G \times_H U$, the quotient space $(G \times U)/H$ for the equivalence relation $(g, y) = (gh^{-1}, hy)$, with $h \in H$, $(g, y) \in G \times U$.*

Now we can state the important *invariant slice theorem* of Palais:

Theorem 2.6 *The tubular neighborhood $\mathcal{U}$ is diffeomorphic to $G \times_H U$.*

Example 2.7 Let us consider the action of the group $O(2)$ in $\mathbb{R}^4 \simeq \mathbb{C}^2$ generated by the following transformations, for any $(z_1, z_2) \in \mathbb{C}^2$:

$$R_\phi(z_1, z_2) = (e^{i\phi} z_1, e^{2i\phi} z_2) \quad (2)$$
$$S(z_1, z_2) = (\overline{z}_1, \overline{z}_2) \quad (3)$$

Representatives of the orbit types are the trivial group (principal stratum), $\mathbb{Z}_2(S)$ (the reflection group which fixes points in the plane $(\Re z_1, \Re z_2)$), $\mathbb{Z}_2(S) \times \mathbb{Z}_2(R_\pi)$ ($\mathbb{Z}_2(R_\pi)$ is the group generated by the rotation of angle π, which fixes points with $z_1 = 0$) and $O(2)$ which only fixes the origin. All orbits out of the origin have the same dimension ($= 1$). However there is a subtle distinction between orbits lying in these three strata. Take a point lying in the principal stratum. Its orbit is the union of two circles. Since the orbit type is trivial, the tubular neighborhood is diffeomorphic to the product $G \times U$ where U is a small neighborhood of x in the normal slice at x. Now take a point lying in the stratum with orbit type $[\mathbb{Z}_2(S) \times \mathbb{Z}_2(R_\pi)]$, e.g. $x = (0, x_2)$ with $x_2 \in \mathbb{R}$. Its orbit is a single circle. An elementary calculation shows that the normal slice at this point is the 3-dimensional (real) subspace generated by elements $(z_1, x_2 + \xi)$, $z_1 \in \mathbb{C}$, $\xi \in \mathbb{R}$. The actions of S and R_π in the normal slice are defined by

$$S \cdot (z_1, x_2 + \xi) = (\overline{z}_1, x_2 + \xi), \quad R_\pi \cdot (z_1, x_2 + \xi) = (-z_1, x_2 + \xi),$$

and we can again find the orbit types for this action. Thanks to the invariant slice theorem, this knowledge is sufficient to fully describe the geometric structure of the action of $O(2)$ in a neighborhood of an orbit of this type. We can proceed in a similar way to analyze the structure near an orbit of type $[\mathbb{Z}_2(S)]$. We will come back to this example in Section 4.3.

We now introduce the concept of orbit space and its basic properties.

Definition 2.8 *The* orbit space V/G *is the quotient of* V *by the equivalence relation* $x \sim y$ *iff* $y \in G \cdot x$.

We said in the Introduction that the orbit space is not in general a manifold. However it inherits the quotient topology, which makes it a Hausdorff space, and the stratified structure of V which was described in the beginning of this section also passes to the orbit space by the canonical projection $o : V \to V/G$. Given a G-invariant function $f : V \to \mathbb{R}$, we can define its projection on the orbit space by setting

$$\tilde{f}(o(x)) = f(x).$$

This allows us to define a "smooth structure" on the orbit space: a function defined on V/G is called C^∞ if its pull-back to V is C^∞.

In the introduction we have seen simple examples where it is easy to identify the stata on the orbit space. In the next section a general way to represent the orbit space will be exposed. An important consequence of the Invariant Slice Theorem about the local structure of the orbit space can be stated now:

Theorem 2.9 *Let* $x \in V$ *and* π *its image in* V/G. *A neighborhood of* π *in the orbit space is diffeomorphic to a neighborhood of* 0 *in* N_x/G_x, *the orbit space of the normal slice at* x *for the action of the isotropy subgroup of* x.

This proposition tells us that if one is interested in the orbit space in a neighborhood only of a point, then it is enough (in general much simpler) to compute the quotient of the normal slice at this point by its isotropy subgroup. We shall see an application in 4.3.

2.2 Invariant polynomials and representation of the orbit space

Let $P_G(V)$ be the ring of G-invariant polynomials in R^n. We recall that invariance means that for any $p \in P_G(V)$, $p(g^{-1}x) = p(x)$ for all $(g, x) \in G \times V$. A classical result of H. Weyl (after Hilbert) asserts that as a ring, $P_G(V)$ is *finitely generated*. This means that there exists a family of invariant polynomials $\pi_1, \dots, \pi_l$ such that, for any $p \in P_G(V)$, there exists a polynomial $\hat{p} \in P(\mathbb{R}^l)$ such that $p(x) = \hat{p}(\pi_1(x), \dots, \pi_l(x))$.
One can always choose a minimal number of elements in this generating family, and moreover these elements can always be chosen as homogeneous polynomials. However even then, the family is not uniquely defined. Moreover its elements are not in general algebraically independent.

Definition 2.10 *We call* Hilbert map *the map* $\pi = (\pi_1, \dots, \pi_l) : V \to \mathbb{R}^l$.

The image Σ of the Hilbert map is a *semi-algebraic* set (a set defined by polynomial equalities and inequalities). It is also, therefore, a stratified set.

Example 2.11 The simplest nontrivial example is that of $\mathbb{Z}_2$ acting on the real line: $x \mapsto -x$ which was discussed in the introduction. $\mathbb{Z}_2$-invariant polynomials contain only even degree terms, therefore the ring of these polynomials is generated by the single element $\pi_1(x) = x^2$. Therefore $\Sigma = \mathbb{R}^+$.

A slightly less straightforward example is when $\mathbb{Z}_2$ acts on the plane $\mathbb{R}^2$ by rotation by π, i.e. by $(x, y) \mapsto (-x, -y)$. Obviously the quadratic monomials $\pi_1(x, y) = x^2$, $\pi_2(x, y) = y^2$ and $\pi_3(x, y) = xy$ are $\mathbb{Z}_2$-invariant. It is easy to check that they also generate the ring of $\mathbb{Z}_2$-invariant polynomials and form a minimal set of generators. However they are not algebraically independent, since $x^2y^2 = (xy)^2$. The relation $\pi_1 \cdot \pi_2 = \pi_3^2$ together with the conditions $\pi_1 \geq 0$, $\pi_2 \geq 0$ defines a cone in $\mathbb{R}^3$. Therefore Σ is the surface of the cone in this case, see Figure 1).

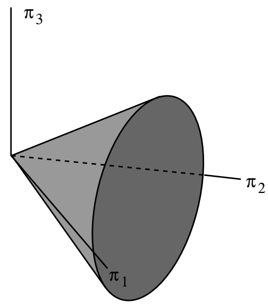

Figure 1: Image of the Hilbert map for the rotation of angle π in the plane.

Another more involved example is when we consider the natural action of D_3, the symmetry group of the equilateral triangle, in the plane. An induction argument combined with some elementary algebraic manipulations shows that the ring of D_3-invariant polynomials is generated by the monomials $\pi_1 = x^2+y^2$ and $\pi_2 = x^3 - 3xy^2$. In this case the generators are algebraically independent. However the image of $\mathbb{R}^2$ under the Hilbert map is defined by the relations $\pi_1 \geq 0$ and $\pi_2^2 \leq \pi_1^3$, which is shown in Figure 2. There are three strata, defined by the following relations:

(i) $\pi_1 > 0$ and $\pi_2^2 < \pi_1^3$;

(ii) $\pi_1 > 0$ and $\pi_2^2 = \pi_1^3$;

(iii) $\pi_1 = 0$.

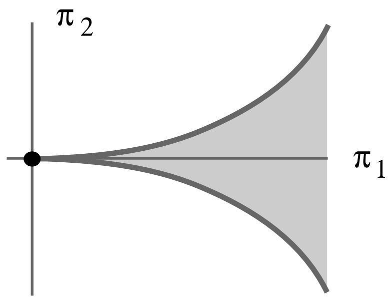

Figure 2: Image of the Hilbert map for the action of D_3 in the plane.

There exists a general method to compute the minimal number of generators of $P_G(V)$, based on the so-called "Molien series" formulae. Once this number is known, it is much easier to determine a family of generators. This method is exposed in Chossat and Lauterbach [7]. In general however, the computation of the generators can be extremely uneasy. For example the number of generators for the action of $SO(3)$ defined by its irreducible representation in the space $\mathbb{R}^{2\ell+1}$ grows very fast with ℓ, hence involving more and more algebraic relations between them. The determination of these generators would require a powerful computer when ℓ is larger than 5 or 6. Nevertheless the existence of a minimal family of generators is of fundamental importance to us, for the following reason.

Theorem 2.12 *Let $\psi : V/G \to \Sigma$ be defined by $\psi(\theta) = \pi(o^{-1}(\theta))$. Then ψ is an homeomorphism.*

It turns out that thanks to a theorem of G. Schwarz [22], not only G-invariant polynomials, but also smooth (i.e. C^∞) G-invariant functions can be expressed in terms of the generators $\pi_1, \dots, \pi_l$. In other words, for any $f \in C^\infty_G(V)$, there exists a function $\tilde{f} \in C^\infty(\mathbb{R}^l)$ such that $f(x) = \tilde{f}(\pi_1(x), \dots, \pi_l(x))$. This defines a smooth structure on the set Σ. Then, by the previous theorem, the orbit space and the image of the Hilbert map are *isomorphic* as stratified sets, together with their smooth structures. It is this identification which we will invoke in order to project smooth equivariant dynamical systems onto the orbit space. For example, a neighborhood of a point in the orbit space identified with Σ can be seen as the intersection of a neighborhood of that point in $\mathbb{R}^l$ intersected with Σ.

In the examples above, the orbit space can therefore be identified with respectively the positive half-line, the positive cone (surface) and the cuspoidal region shown on Figure 2.
From now on, we shall, by abuse of language, call "orbit space" the set S itself.

3 The orbit space reduction of equivariant dynamical systems

3.1 Equivariant dynamics and their projection on the orbit space

Let X be a G-equivariant smooth (C^∞) vector field in V, $G \subset O(n)$. The following proposition shows the basic consequences of the equivariance.

Proposition 3.1 *(i) If $\phi(x,t)$ is the flow defined by X, then*
$\phi(g \cdot x, t) = g \cdot \phi(x,t)$ for all $g \in G$, $(x,t) \in V \times \mathbb{R}$.
(ii) Let S be the stratum of some orbit type. If $x \in S$, then $\phi(x,t) \in S$ for all $t \in \mathbb{R}$.

Of course, the closure of a trajectory may contain points in the boundary of S. Point (i) tells us that the dynamics induced by the vector field is itself equivariant. it follows that we can project this dynamics (the flow) on the orbit space by setting

$$\tilde{\phi}(\pi(x), t) = \pi \circ \phi(x,t). \tag{4}$$

The map $\tilde{\phi}$ has the same group property as ϕ:

$$\tilde{\phi}(\pi, t+t') = \tilde{\phi}(\tilde{\phi}(\pi, t), t') \text{ and } \tilde{\phi}(\pi, 0) = \pi.$$

Given a $\tilde{\phi}$-invariant set A in the orbit space, its inverse image in V is invariant under the flow of X. There is one case of special interest:

Definition 3.2 *If A is a point (an equilibrium for $\tilde{\phi}$), we call $\pi^{-1}(A)$ a* relative equilibrium*.*

The inverse image of a point is a G-orbit. Therefore a relative equilibrium is a G-orbit to which the vector field is tangent at any point. What is the dynamics on a relative equilibrium? This is an information that the knowledge of A alone would not provide. In fact it is not very difficult to show that any trajectory is dense in a torus of dimension at most equal to the dimension of rank of the group $N_G(G_x)/G_x$, where x is some point on the relative equilibrium (see Chossat and Lauterbach [7] for details).
Similarly, we have the following definition:

Definition 3.3 *If A is a periodic orbit for $\tilde{\phi}$, then $\pi^{-1}(A)$ is called a* relative periodic orbit*.*

Trajectories on relative periodic orbits are also dense on tori, however their classification is harder than that of relative equilibria and we shall not pursue on this question (see Field [11], Krupa [18] and Chossat and Lauterbach [7]). We can extend to the projected dynamics most usual notions of dynamical systems theory. For example, let π_0 be an equilibrium in the orbit space.

Definition 3.4 *The set of points $\pi \in S$ such that $\lim_{t\to+\infty}$ (resp. $\lim_{t\to-\infty}$) of $\tilde{\phi}(\pi, t)$ is π_0, is called the V/G-*stable *(resp. V/G-*unstable*) manifold of π_0.*

The V/G-stable and V/G-unstable manifolds are not true manifolds in general, however their restrictions to strata are smooth manifolds. If there exists a neighborhood of π_0 (in S) such that any point in this neighborhood belongs to the G-stable (resp. G-unstable) manifold, then the pull-back of this equilibrium is an *orbitally stable* (resp. orbitally unstable) relative equilibrium.
The next step will be to associate with the flow $\tilde{\phi}$ a vector field on the orbit space.

3.2 Projecting equivariant vector fields: computation and properties

Let $\{\pi_1, \ldots, \pi_l\}$ be a (minimal) family of generators of the ring of G-invariant polynomials. Then

$$\frac{d}{dt}\pi_i(x) = D_x\pi_i(x) \cdot \dot{x} = \langle \mathrm{grad}\pi_i(x), \dot{x} \rangle,$$

where the last term is the dual product. Now, the gradient of a G-invariant function is a G-equivariant mapping (straightforward), and $\dot{x} = X(x)$ which is also G-equivariant by hypothesis. Since G is a group of isometries, it immediately follows that the last term in the above expression is a G-invariant smooth function. By Schwarz's theorem evoked in the previous section, it follows that this term is in fact a function of the π_j's: there exists a smooth function $\tilde{X}_i : \mathbb{R}^l \to \mathbb{R}$ such that we can write

$$\dot{\pi}_i = \tilde{X}_i(\pi_1, \ldots, \pi_l), \quad i = 1, \ldots, l \tag{5}$$

and this function is defined by the formula

$$\tilde{X}_i(\pi_1(x), \ldots, \pi_l(x)) = \langle \mathrm{grad}\pi_i(x), X(x) \rangle$$

where brackets indicate the G-invariant inner product in V. Now, let $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_l)$. This defines a smooth map in $\mathbb{R}^l$, hence a smooth vector field in that space. Moreover, by construction, given any point $(\pi_1, \ldots, \pi_l) \in S$, the vector $\tilde{X}(\pi_1, \ldots, \pi_l)$ is *tangent to the stratum* in which $(\pi_1, \ldots, \pi_l)$ lies. The restriction of $\tilde{X}$ to S is therefore well-defined and is the projection of the vector field X to that space.

Proposition 3.5 *The flow of $\tilde{X}$ in S coincides with the projected flow $\tilde{\phi}(\pi, t)$ defined in the previous section.*

The proof of this proposition is straightforward algebra.
It still remains to understand how the flow of $\tilde{X}$ restricts to S. indeed, $\tilde{X}$ is defined and smooth in $\mathbb{R}^l$, however S is a singular space unless the action of the group is free. This can produce strange phenomena, as the examples below show.

Example 3.6 Consider the linear vector field X defined by the system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cy. \end{aligned}$$

It is equivariant by the action of $\mathbb{Z}_2$ defined by $(x, y) \mapsto (-x, -y)$. We have seen that the orbit space can be defined as the image of the map

$$(x, y) \mapsto (\pi_1 = x^2, \pi_2 = y^2, \pi_3 = xy).$$

A straightforward calculation shows that the projected $\tilde{X}$ yields the system in $\mathbb{R}^3$

$$\begin{aligned} \dot{\pi}_1 &= 2a\pi_1 + 2b\pi_3 \\ \dot{\pi}_2 &= 2c\pi_2 \\ \dot{\pi}_3 &= b\pi_2 + (a + c)\pi_3. \end{aligned}$$

If $a = -c \neq 0$, 0 is a hyperbolic equilibrium of X, while $\tilde{X}$ has a zero eigenvalue at the origin in $\mathbb{R}^3$. However, as we shall see below, this is a "fake" eigenvalue for the dynamics on the orbit space.

As an other example, consider now the vector field defined by

$$\begin{aligned} \dot{x} &= x^2 - y^2 \\ \dot{y} &= -2xy. \end{aligned}$$

This system is D_3-equivariant for the action defined in the example in the previous section. Moreover this does not contain any linear term. The projection of this vector field yields the system

$$\begin{aligned} \dot{\pi}_1 &= \pi_2 \\ \dot{\pi}_2 &= 6\pi_1^2 \end{aligned}$$

which has a non-zero linear part.

More is therefore needed in order to characterize the dynamics near an equilibrium in the orbit space. This problem was tackled by Koenig [17] who produced several new results, some of which we now state.

Theorem 3.7 *Let $X(x) = L_s x + L_n x + R(x)$ be a G-equivariant vector field in V, where L_s is the semisimple linear part of X, L_n is the nilpotent linear part and $R(x) = o(x)$. Similarly let $\tilde{X}(\pi) = \tilde{L}_s\pi + \tilde{L}_n\pi + \tilde{R}(\pi)$ be the projected vector field on the orbit space. Note that $X(0) = 0$ implies $\tilde{X}(0) = 0$. Then there exists a choice of generators $\pi_1, \ldots, \pi_l$ such that:*

(i) $\tilde{L}_s$ is the projection of L_s and is semisimple;

(ii) $\tilde{L}_n\pi + \tilde{R}(\pi)$ is the projection of $L_n x + R(x)$;

(iii) The eigenvalues of $\tilde{L}_s$ are linear combinations with positive interger coefficients of the eigenvalues of L_s.

If 0 is a hyperbolic equilibrium of X, how can we characterize this on the orbit space? We first need to introduce a new concept.

Definition 3.8 *For $\pi \in S$, let $[\pi]$ be the real half-line in $\mathbb{R}^l$ containing 0 and π. The* tangent cone *to S at 0, $C_0(S)$, is the subset of $\mathbb{R}^l$ given by the limits of the half-lines $[\pi]$ when $\pi \to 0$ in S.*

The computation of the tangent cone is easy in general, for it is identical to the tangent cone of the image in $\mathbb{R}^l$ of the invariant generators which have degree less than or equal to two.

Example 3.9 For the $\mathbb{Z}_2$ action in the plane by rotation of angle π, the tangent cone is S itself (a cone in $\mathbb{R}^3$, as we already saw).
For the action of D_3 already discussed above, the tangent cone is the semi-axis $\pi_1 \geq 0$ (see Figure 2).

Let E_+, E_- and E_0 denote respectively the linear stable, unstable and center manifolds of $\tilde{X}$ in V. We suppose that $\tilde{X}(0) = 0$.

Definition 3.10 *The equilibrium at 0 is V/G-hyperbolic if $E_0 \cap C_0(S) = \{0\}$. It is V/G-stable if $E_- \cap C_0(S) = C_0(S)$ and V/G-unstable if $E_+ \cap C_0(S) = C_0(S)$.*

Theorem 3.11 *0 is a V/G-hyperbolic (resp. V/G-stable, V/G-unstable) equilibrium of $\tilde{X}$ iff 0 is a hyperbolic (resp. stable, unstable) equilibrium of X.*

Finally, let us state two classical results in ODE's which pass to the orbit space in the case of an equivariant vector field.

1. The equivariant version of the theorem of Grobman and Hartman (see Chossat and Lauterbach [5, 7]):

   Theorem 3.12 *Let X be a G-equivariant smooth vector field such that $X(0) = 0$, and $\tilde{L} = D_\pi \tilde{X}(0)$. If 0 is V/G-hyperbolic in $\mathbb{R}^l$, then there exists a neighborhood of 0 in $\mathbb{R}^l$ such that in this neighborhood, the flow of $\tilde{X}$ is topologically conjugated to the flow of $\tilde{L}$.*

   The proof is based on the fact that the conjugacy between X and $L = DX(0)$ in V can be made G-equivariant.

2. The Poincaré-Bendixson theorem:

   Theorem 3.13 *Let S be a two-dimensional stratum in V/G and $\tilde{X}$ be the projection of an equivariant vector field. Suppose that S contains a minimal invariant set M of $\tilde{X}$, i.e. a compact flow-invariant set that cannot be decomposed into smaller invariant sets. Then M is either an equilibrium, or a periodic orbit, or a torus.*

   This theorem follows quite directly from the theorem stated by Hartman [16] for vector fields on smooth 2-manifolds, given that strata are smooth manifolds in $\mathbb{R}^l$.

Remark 3.14 The above definitions and theorems about the local behaviour of the flow near an equilibrium at 0, apply in fact to *any* relative equilibrium, thanks to Proposition 2.9. Indeed, it allows us to identify a neighborhood of the projection on the orbit space of a relative equilibrium $G \cdot x$, with a neighborhood of 0 in the orbit space N_x/G_x where N_x is the normal slice at x. Now, thanks to a theorem of Field [11] and Krupa [18], we can decompose the vector field into a "normal" component X_N in N_x and a "tangent" vector field which describes the "drift" along the continuous action of the group. The relevant dynamics is all contained in X_N, which has equilibrium 0 and which we may project onto N_x/G_x. The explicit construction of X_N can be found in Chossat and Lauterbach [7]. An example is provided in section 4.3.

4 Applications: three examples

The following three problems are chosen to illustrate the orbit space reduction method because (a) they are, in a sense, classical and their treatment by other methods can easily be found in the literature, (b) the calculations are simple enough to not obscure the purpose.
Other examples may be found in Chossat and Lauterbach [7] and bibliography therein.

4.1 Mode interaction of travelling waves

We consider a smooth function $H : \mathbb{R}^4 \to \mathbb{R}$ which is invariant under the following action of the rotation group $SO(2)$: we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and we set

$$R_\varphi(z_1, z_2) = (e^{i\varphi} z_1, e^{2i\varphi} z_2), \quad \varphi \in S^1.$$

This action is often called a 1-2 mode interaction.
In the complex variables, H is a function of $(z_1, z_2, \bar{z}_1, \bar{z}_2)$. It is well-known that a minimal family of generators for this action is

$$\pi_1 = z_1\bar{z}_1, \quad \pi_2 = z_2\bar{z}_2, \quad \pi_3 = \mathrm{Re}\{z_1^2\bar{z}_2\}, \quad \pi_4 = \mathrm{Im}\{z_1^2\bar{z}_2\}.$$

Note that $\pi_1^2\pi_2 = \pi_3^2 + \pi_4^2$. Moreover we assume that $H(0,0,0,0) = 0$, so that

$$H(z_1, z_2, \bar{z}_1, \bar{z}_2) = a\pi_1 + b\pi_2 + c\pi_3 + d\pi_4 \tag{6}$$

where a, b, c and d are smooth functions of $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$'s. Finally we also assume that $a(0) = 1$ and $b(0) = 2 + \mu$, μ close to 0 (detuning parameter from exact 1:2 resonance).

The equivariant Hamiltonian vector field derived from H is defined by

$$\dot{z}_1 = i\frac{\partial H}{\partial \bar{z}_1} \tag{7}$$

$$\dot{z}_2 = i\frac{\partial H}{\partial \bar{z}_2}. \tag{8}$$

and we aim to analyze the flow when z_1 and z_2 are small.
This system originates from the study of capillary-gravity travelling waves propagating at the interface of two fluid in a given direction x. In the case considered here, the surface elevation of the interface is, at first order, given by the expression

$$\eta(x,t) = 2\mathrm{Re}\{z_1 e^{i(t-x)}\} + 2\mathrm{Re}\{z_2 e^{2i(t-x)}\},$$

a situation which can indeed occur when capillarity effects are taken into account in addition to gravity. It is not the purpose of this paper to develop on the details of motivating problems and we let it to the reader to look at the specific bibliography (Christodoulides and Dias [9] contains an extended review). Let us simply notice that (a) the derivation of this system is only approximate; (b) by allowing waves travelling in the opposite direction, that is, by taking into account the reflection symmetry on the real line, we would multiply the phase space dimension by 2 and render the problem more complicated but also richer (this more general approach has been investigated by Bridges [2], and Chossat and Dias [4]); (c) other mechanical systems can be described by this type of Hamiltonian, like the double spherical pendulum, see Rott [20]; (d) in the capillary-gravity waves problem, there is an additional symmetry by time reversibility, however we will not use it to solve the Hamiltonian system in this case.

Thanks to the Hamiltonian structure of the vector field, there are two conserved quantities: the Hamiltonian H (energy) and the angular momentum $J = \pi_1 + 2\pi_2$, associated with the $SO(2)$ invariance (Noether's theorem). These two functions being $SO(2)$ invariant, they can be projected to the orbit space, and this allows us to elegantly resolve the problem (see Cushman and Bates [10] for a general exposition of orbit space reduction in the Hamiltonian context).

According to previous sections, the orbit space can be identified with the semi-algebraic subset of $\mathbb{R}^4$:

$$S = \{\pi \mid \pi_1 \geq 0, \pi_2 \geq 0, \pi_1^2\pi_2 = \pi_3^2 + \pi_4^2\}.$$

There are three strata: the origin (fixed by $SO(2)$), the axis $\pi_1 = 0$ (fixed by $\varphi = \pi$) and the principal stratum (trivial isotropy). Conservation of momentum reads $\pi_2 = \frac{1}{2}(J - \pi_1)$, $J \geq 0$ fixed. This allows us to eleminate the variable π_2, hence reducing the problem to the dynamics in the space

$$S = \{(\pi_1, \pi_3, \pi_4) \in \mathbb{R}^3 \mid \pi_1 \geq 0, \pi_1^2(J - \pi_1) = 2(\pi_3^2 + \pi_4^2)\}. \tag{9}$$

This is a semi-algebraic surface which has the form of a "pear" (pinched sphere at the origin O) with symmetry axis $O\pi_1$. The non principal strata have been reduced to the origin (i.e. $\pi_1 = 0$) which corresponds to the stratum of type $\mathbb{Z}_2$ if $\pi_2 > 0$ and to the stratum of type $SO(2)$ if $\pi_2 = 0$. The origin being an isolated point in a stratum, it is necessarily an equilibrium for the projected flow (this was an observation made by Michel [19] thirty years ago). Its pull-back in the z_j's phase space is a circle, hence a periodic orbit (of period close to 2π) which corresponds to a "pure mode" travelling wave with spatial period π.
We now take advantage of the second conserved quantity. By hypothesis, $H = J + E$, where

$$E = \mu(J - \pi_1) + \alpha\pi_1^2 + \beta\pi_1(J - \pi_1) + \gamma(J - \pi_1)^2 + c(0)\pi_3 + d(0)\pi_4 + h.o.t.$$

is also a first integral of the flow (the coefficients α, β, γ depend on the Taylor expansion of H). Assuming that $c(0)$ and $d(0)$ are not simultaneously equal to zero, and recalling that the π_j's are close to 0, one observes that the level surfaces are leaves which intersect the reduced orbit space when E belongs to a certain interval $[E_1, E_2]$. In the limits $E = E_1$ and $E = E_2$, the leaves are tangent at one point to the orbit space. These two points are therefore equilibria which correspond after pull-back to travelling waves which mix the z_1 and z_2 modes and moreover have frequency equal to twice the frequency of the "pure mode" travelling waves (they correspond, in the wave problem, to the so-called "Wilton's ripples"). While E is varied between these two limits, the leaves intersect the reduced orbit space along closed curves on which the flow is periodic, except for the curve which passes by the origin. Since the origin is an equilibrium, this curve defines a homoclinic orbit in the reduced orbit space. The lifts of the periodic orbits are relative periodic orbits with two frequencies (they wind up along 2-tori). The lift of the homoclinic orbit is a 1-parameter family of homoclinic trajectories to the "pure mode" travelling waves.

4.2 Hopf bifurcation with D_4 symmetry

This problem, which arises notably in the case of a system formed by four coupled oscillators on a ring, has been studied in details by Swift [23]. Various interesting phenomena have been described in this problem, such as the generic (in the sense of "occuring in an open region of parameter space") bifurcation of periodic orbits with minimal symmetry or even of invariant tori. The system under consideration is the normal form for Hopf bifurcation in $V = \mathbb{R}^4 \simeq \mathbb{C}^2$, where the action of $G = D_4 \times S^1$ is defined as follows:

$$\rho \;:\; (z_1, z_2) \mapsto (iz_1, -iz_2), \quad \mu : (z_1, z_2) \mapsto (z_2, z_1) \tag{10}$$

$$R_\psi \;:\; (z_1, z_2) \mapsto (e^{i\psi} z_1, e^{i\psi} z_2) \tag{11}$$

The method of Swift consisted in projecting the system on the S^1 orbit space identified with $\mathbb{R}^3$ after a suitable choice of coordinates. Here we simply aim to

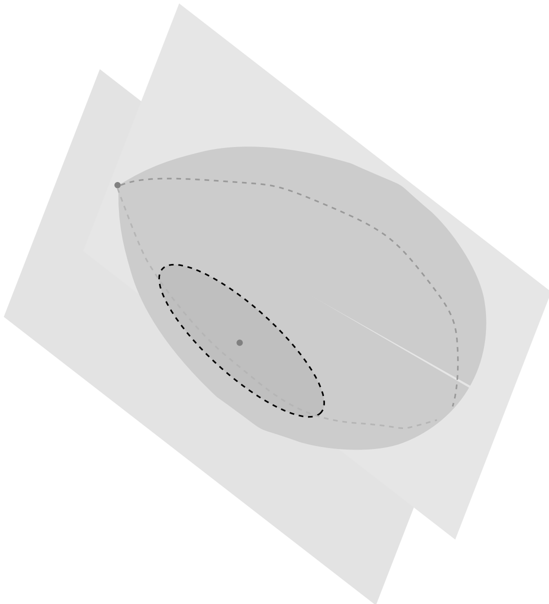

Figure 3: The reduced orbit space for the $1-2$ mode interaction and $1:2$ resonance of travelling waves. Picture shows the "pear" intersected by two sheets $H = const$: along a periodic orbit and along a homoclinic orbit to the equilibrium at the origin.

show that the "full" $D_4 \times S^1$ orbit space reduction also leads to an amenable system on which geometric insight can help in describing the bifurcations and dynamics.

A family of generators for the ring $P_G(V)$ is given by

$$\pi_0 = z_1\bar{z}_1 + z_2\bar{z}_2, \quad \pi_1 = z_1\bar{z}_1 z_2\bar{z}_2, \tag{12}$$
$$\pi_2 = (z_1\bar{z}_2)^2 + (z_2\bar{z}_1)^2, \quad \pi_3 = i(z_1\bar{z}_1 - z_2\bar{z}_2)[(z_1\bar{z}_2)^2 - (z_2\bar{z}_1)^2]. \tag{13}$$

The proof can be obtained by the method exposed in Chossat and Lauterbach [7] (computation of the minimal number of generators by using Molien series) or by a direct computation and induction argument. These generators satisfy the algebraic relation

$$\pi_3^2 = (4\pi_1 - \pi_0^2)(\pi_2^2 - 4\pi_1^2) \tag{14}$$

and the identification of the orbit space with a "3-dimensional" subset of $\mathbb{R}^4$ follows straightforwardly. Given this relation, the orbit types are

- D_4: stratum $\{\pi_j = 0, \, j = 0, \ldots, 3\}$;
- $\mathbb{Z}_4 \simeq \langle \rho \cdot \mathbb{R}_{-\pi/4} \rangle$: stratum $\{\pi_1 = \pi_2 = \pi_3 = 0\}$, $\pi_0 > 0$;
- $\mathbb{Z}_2 \simeq \langle \mu \rangle$: stratum $\{4\pi_1 = 2\pi_2 = \pi_0^2, \, \pi_0 > 0\}$;
- $\mathbb{Z}_2 \simeq \langle \rho \cdot \mu \rangle$: stratum $\{4\pi_1 = -2\pi_2 = \pi_0^2, \, \pi_0 > 0\}$;
- the principal stratum.

The G-equivariant vector field truncated at order three reads, in the (z_1, z_2) space and after a normalization of the coefficients,

$$\begin{aligned}\dot{z}_1 &= z_1(\lambda + i\omega + A\pi_0 + Bz_1\bar{z}_1) + C\bar{z}_1 z_2^2 \\ \dot{z}_2 &= z_2(\lambda + i\omega + A\pi_0 + Bz_2\bar{z}_2) + Cz_1^2\bar{z}_2,\end{aligned}$$

where the coefficients A,B and C are complex, $\omega \in \mathbb{R}$ and λ is the bifurcation parameter. A calculation shows that the projected vector field has the form

$$\begin{aligned}\dot{\pi}_0 &= 2(\lambda + A_r\pi_0)\pi_0 + 2B_r(\pi_0^2 - 2\pi_1) + 2C_r\pi_2 \\ \dot{\pi}_1 &= 4(\lambda + A_r\pi_0)\pi_1 + 2B_r\pi_0\pi_1 + C_r\pi_0\pi_2 + C_i\pi_3 \\ \dot{\pi}_2 &= 4(\lambda + A_r\pi_0)\pi_2 + 2B_r\pi_0\pi_2 + 4C_r\pi_0\pi_1 + 2B_i\pi_3 \\ \dot{\pi}_3 &= 6(\lambda + A_r\pi_0)\pi_3 + 4B_r\pi_0\pi_3 + 2C_i(\pi_2^2 - 4\pi_1^2) + 2(2C_i\pi_1 - B_i\pi_2)(\pi_0^2 - 4\pi_1).\end{aligned}$$

This system looks complicated, but it has some nice features.
Let us scale the variables in a way compatible with the orbit space structure, namely:

$$\pi_1 = \pi_0^2 x_1, \quad \pi_2 = \pi_0^2 x_2, \quad \pi_3 = \pi_0^3 x_3, .$$

Then the projected system can be re-written as

$$\begin{aligned}\dot{\pi}_0 &= 2(\lambda + A_r\pi_0)\pi_0 + 2B_r\pi_0^2(1 - 2x_1) + 2C_r\pi_0^2 x_2 \\ \dot{x}_1 &= \pi_0[2B_r x_1(4x_1 - 1) + C_r x_2 - 4C_r x_1 x_2 + C_i x_3] \\ \dot{x}_2 &= \pi_0[2B_r x_2(4x_1 - 1) + 4C_r x_1 - 4C_r x_2^2 + 2B_i x_3] \\ \dot{x}_3 &= \pi_0[2B_r x_3(6x_1 - 1) - 6C_r x_2 x_3 + 2C_i(x_2^2 - 4x_1^2) + \\ &\quad 2(2C_i x_1 - B_i x_2)(1 - 4x_1)]\end{aligned}$$

The remarkable fact is that solving this system is equivalent to solving first the decoupled system

$$\begin{aligned}\frac{dx_1}{ds} &= (2B_r x_1 - C_r x_2)(4x_1 - 1) + C_i x_3 \\ \frac{dx_2}{ds} &= 2B_r x_2(4x_1 - 1) + 4C_r(x_1 - x_2^2) + 2B_i x_3 \\ \frac{dx_3}{ds} &= 2B_r x_3(6x_1 - 1) - 6C_r x_2 x_3 + \\ &\quad 2C_i(x_2^2 - 4x_1^2) + 2(2C_i x_1 - B_i x_2)(1 - 4x_1),\end{aligned} \tag{15}$$

then solving the system

$$\dot{\pi}_0 = 2\lambda\pi_0 + \pi_0^2[A_r + 2B_r(1 - 2x_1) + 2C_r x_2] \tag{16}$$

$$\dot{s} = \pi_0. \tag{17}$$

This decomposition was found by Swift [23] with a different kind of coordinates, and we will simply follow his arguments. Clearly, if an equilibrium is found to the system (15), a corresponding equilibrium

$$\pi_0 = \frac{-2\lambda}{A_r + 2B_r(1 - 2x_1) + 2C_r x_2}$$

is obtained for (16), provided the right hand side is positive. Moreover, let us assume that (15) has a periodic orbit and that on this orbit, the function $A_r + 2B_r(1 - 2x_1) + 2C_r x_2$ doesn't change sign. We may assume that this sign

is negative for definiteness. Then it is not difficult to show that the system (16)-(17) has a unique periodic solution for $\lambda > 0$. In Swift's context, this observation has been generalized by Field [12] to what is known as the "Invariant Sphere Theorem". It can be stated for systems in the orbit space, however we shall not use it here.
The domain in $\mathbb{R}^3$ in which the system (15) holds is the *reduced orbit space*, which is defined by the relations

$$x_3^2 = (4x_1 - 1)(x_2^2 - 4x_1^2), \quad x_1 \geq 0.$$

Let us first look for branches of equilibria of (15). It is *sufficient* for this to solve the equations $\dot{x}_1 = \dot{x}_2 = 0$ together with the condition (14). The equation $\dot{\pi}_3 = 0$ will indeed be automatically satisfied.
The solution with $x_1 = x_2 = x_3 = 0$ corresponds to an equilibrium with isotropy $\mathbb{Z}_4$. The solutions with $x_3 = 0$, $4x_1 = 1$ and $2x_2 = \pm 1$ belong to the two strata whose orbit type is isomorphic to $\mathbb{Z}_2$. Let us look for other solutions. Some additional algebra combining these equations with the relation (14) lead to the relations

$$\begin{aligned} |C|^2 x_2 &= 2\text{Re}\{B\bar{C}\}x_1 \\ 2(\text{Im}\{B\bar{C}\})^2 x_1 &= (|B|^2 - |C|^2)|C|^2. \end{aligned}$$

Note that the definition of the orbit space imposes the condition $x_2^2 > 4x_1^2$. Replacing this in the first of these equations, we obtain the condition $|\text{Re}\{B\bar{C}\}| < |C|^2$. Then the second equation is solvable iff $|B|^2 > |C|^2$. These are precisely the conditions found by Swift for the existence of periodic orbits with minimal symmetry.

We now show that branches of periodic solutions can also bifurcate generically (for an open set of parameter values). This was the main result of Swift [23] where more informations can be found about existence and stability of these branches. We consider the case when $|B|^2 < |C|^2$. This insures that no equilibrium can bifurcate in the principal stratum.
It is a simple exercise to check that if $B_r > 3|C_r|$, then the two $\mathbb{Z}_2$ equilibria are foci in the reduced orbit space while the $\mathbb{Z}_4$ equilibrium is a saddle. The reduced orbit space looks like a "croissant", as shown in Figure 4. It is a compact invariant set, and under the foregoing conditions, the Poincaré -Bendixson theorem applies to insure the existence of (at least) one periodic orbit in the principal stratum (figure 4).

The equilibria and periodic orbits that have been found to bifurcate in the orbit space correspond respectively to relative equilibria (in fact, periodic orbits with a non-vanishing frequency at bifurcation, due to the Hopf bifurcation hypothesis) and to relative periodic orbits (two-tori on which trajectories show no frequency locking). The persistence of these branches of solutions when higher order terms are added to the normal form is easy to show under non-degeneracy conditions. The fact that for the original D_4-equivariant vector field, branches of periodic orbits and of invariant tori persist when the non S^1-equivariant rest is added to the normal form, is more subtle but can also be treated rigorously (see Field [13]).

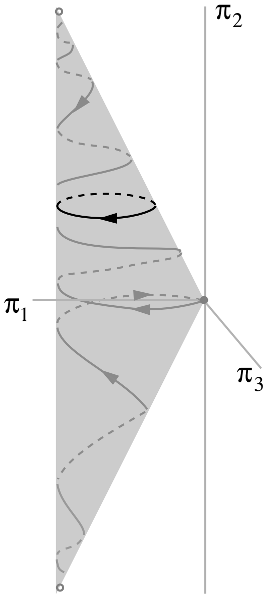

Figure 4: The reduced orbit space for Hopf bifurcation with $D_4 \times S^1$ symmetry and a typical configuration of the dynamics when a periodic orbit exists.

4.3 Resonant bifurcation of a homoclinic cycle in an $O(2)$ symmetric system

We consider the action of the group $O(2)$ in $\mathbb{R}^4 \simeq \mathbb{C}^2$ defined as in example 2.7:

$$R_\varphi(z_1, z_2) = (e^{i\varphi} z_1, e^{2i\varphi} z_2), \quad S(z_1, z_2) = (\bar{z}_1, \bar{z}_2).$$

This is a well-known fact that vector fields which are equivariant by this action can possess a robust homoclinic cycle, by which we mean a set of heteroclinic connections between equilibria which belong to the *same* group orbit, and which moreover are not destroyed by small equivariant perturbations of the vector field. We will see below how these homoclinic cycles are built. Their existence and stability was thoroughly analyzed through a bifurcation analysis of the codimension 2 singularity at the origin by Armbruster, Guckenheimer and Holmes [1]. In particular it was proven that if we denote by λ_- and λ_+ the contracting and expanding eigenvalues at an equilibrium along the heteroclinic trajectory, then the following is true: (i) if $|\lambda_-| > \lambda_+$, the invariant set formed by the orbit of equilibria and their heteroclinic connections is asymptotically stable; (ii) under generic conditions, if $|\lambda_-|$ crosses λ_+ as parameters are varied, then a family of relative periodic orbits (”modulated travelling waves”) bifurcates from the homoclinic cycle. Our aim in this example is to perform the same study, however not in the codimension 2 bifurcation problem but instead starting from a vector field with a given homoclinic cycle. We will make use of different tools, in particular of the local version of the orbit space reduction which we will apply to the so-called ”resonant bifurcation” analyzed by Chow, Deng and Fiedler [8] in the case of homoclinic orbits of non symmetric vector fields.

A minimal family of generators for the invariant polynomials is given by

$$\pi_1 = z_1\bar{z}_1, \quad \pi_2 = z_2\bar{z}_2, \quad \pi_3 = \frac{1}{2}(z_1^2\bar{z}_2 + \bar{z}_1^2 z_2).$$

The orbit space is defined in $\mathbb{R}^3$ by the relations

$$\pi_1 \geq 0, \quad \pi_2 \geq 0, \quad \pi_3^2 \leq \pi_1^2\pi_2.$$

The orbit types of this action of $O(2)$ are then

- $O(2)$, stratum $\{0\}$;
- $\mathbb{Z}_2^2$ (generated by R_π and S), stratum $\pi_1 = 0$, $\pi_2 > 0$;
- $\mathbb{Z}_2 \simeq \{Id, S\}$, stratum S defined by $\pi_1 > 0, \quad \pi_3^2 = \pi_1^2\pi_2$;
- $\{Id\}$, the principal stratum.

The orbit space is a kind of solid cone, the boundary of which contains all non principal strata (figure 5).

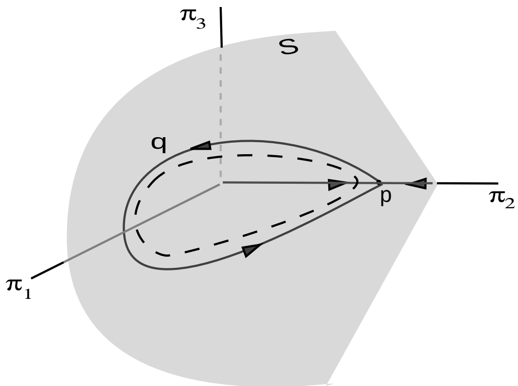

Figure 5: The orbit space for the 1-2 mode interaction with $O(2)$ symmetry, showing the image of a robust homoclinic cycle and in dotted line a bifurcated periodic orbit. The orbit $q(t)$ lies on the boundary S while the periodic orbit lies in the principal stratum (interior of the "cone").

Let p be an equilibrium for the projected vector field, lying on the axis $\pi_2 > 0$. Suppose that on the boundary of the orbit space, a trajectory $q(t)$ is homoclinic to p (figure 5). Then this orbit is robust under small perturbations of the projected dynamics, because (a) it is constrained to lie on the invariant surface S and (b) p is a sink on one of the two folds of this surface. This homoclinic orbit is the projection of a robust homoclinic cycle connecting equilibria of opposite sign in the (z_1, z_2) space, and we refer to Armbruster et al [1] for further details about its structure.

Our purpose is to analyze the dynamics close to the homoclinic orbit. When is the invariant set formed by the equilibrium and its homoclinic orbit an attractor, and if a change occurs in stability as a parameter is varied, what kind of bifurcation is that?

In order to achieve this goal we could compute the projected vector field and work on it directly, like in Chossat [3]. However it is more amenable to project the vector field on the local orbit space at p, because the dynamics near the invariant set is basically controlled by the linearized flow at p.
The inverse image of p is the circle $C_p = \{(0, z_2),\ |z_2| = p\}$. We choose the point $P = (0, \hat{x}_2)$, $\hat{x}_2 \in \mathbb{R}$, on this orbit. The normal slice at P is the hyperplane $N = \{(z_1, \hat{x}_2 + \xi),\ z_1 \in \mathbb{C}, \xi \in \mathbb{R}\}$. We now write $z_1 = x_1 + iy_1$. The isotropy group of P is $\mathbb{Z}_2^2$ generated by R_π and S, which act as follows in N:

$$R_\pi(x_1, y_1, \xi) = (-x_1, -y_1, \xi), \quad S(x_1, y_1, \xi) = (x_1, -y_1, \xi). \tag{18}$$

It is easily found that

$$N/\mathbb{Z}_2^2 \simeq \{(u = x_1^2, v = y_1^2, w = \xi \mid u \geq 0, v \geq 0)\}.$$

The steps are now the following: 1) determine the normal component of the vector field from Krupa's decomposition, see Remark 3.14, 2) project this component on the local orbit space.
The $O(2)$-equivariant, smooth vector fields have the following general structure (see Chossat [3]):

$$\begin{aligned} \dot{z}_1 &= az_1 + b\bar{z}_1 z_2 \\ \dot{z}_2 &= cz_2 + dz_1^2 \end{aligned}$$

where a,, b, c and d are smooth functions of (π_1, π_2, π_3). We compute the normal vector field by looking for solutions of the form $(e^{i\varphi(t)}z_1, e^{2i\varphi(t)}(\hat{x}_2 + w))$, $\phi(t) \in \mathbb{R}$. By $SO(2)$ equivariance the equations become

$$\begin{aligned} i\dot{\varphi}z_1 + \dot{z}_1 &= az_1 + b\bar{z}_1(\hat{x}_2 + \xi) \\ 2i\dot{\varphi}(\hat{x}_2 + \xi) + \dot{\xi} &= c(\hat{x}_2 + \xi) + dz_1^2. \end{aligned}$$

Let us write again $z_1 = x_1 + iy_1$. The imaginary part of the second equation reads

$$2\dot{\varphi}(\hat{x}_2 + \xi) = 2dx_1y_1$$

and can be readily solved for $\dot{\varphi}$ since $\hat{x}_2 \neq 0$. Replacing $\dot{\varphi}$ by this solution in the other equations gives us a system in (x_1, y_1, ξ) which is the normal vector field.
We are essentially interested in the linearized flow near 0 in $N/\mathbb{Z}_2^2$. A simple calculation shows that this is the flow of the system (not forgetting that the origin is by assumption an equilibrium)

$$\begin{aligned} \dot{u} &= 2(a + b\hat{x}_2)u \\ \dot{v} &= 2(a - b\hat{x}_2)v \\ \dot{w} &= 2\hat{x}_2^2 \frac{\partial c}{\partial \pi_2} w, \end{aligned}$$

where the functions a, b and $\partial c/\partial \pi_2$ are evaluated at $\pi_1 = 0$, $\pi_2 = \hat{x}_2^2$ and $\pi_3 = 0$.

Remark 4.1 *It should be emphasized that in this simple example, there is no need to project further the normal vector field on the local orbit space. However* *there are examples where this projection really helps organizing the calculations (see Chossat, Guyard and Lauterbach[6]). Our purpose being to illustrate a method, we nevertheless proceed to this projection. The reader can easily check by himself that this further projection does make things neither simpler nor more complicated...*

In accordance with the existence of a robust homoclinic orbit and to fix the ideas, we can assume that the eigenvalues are such that $\lambda_- = a + b\hat{x}_2 < 0$, $\lambda_+ = a - b\hat{x}_2 > 0$ and $\lambda_r = 2\hat{x}_2^2 \partial c/\partial \pi_2 < 0$. We also know that the planes $u = 0$ and $v = 0$ are flow invariant because they correspond to the orbit types $\mathbb{Z}_2(S)$ and $\mathbb{Z}_2(SR_\pi)$ (off the w axis). Notice that in the global orbit space, they correspond (locally) to the two folds of the *same* stratum of type $\mathbb{Z}_2$. The forward part of the homoclinic orbit q lies in the plane $v = 0$ while the backward part lies in the plane $u = 0$.
It follows from the equivariant version of the Grobman-Hartman theorem that the flow in a neighborhood of 0 in the local orbit space is "topologically conjugated" to its linearization. In other words we may, by a continuous change of variables, replace the projected vector filed by its linear part in an neighborhood of 0. This change of variables can be made C^1 or smoother under additional non-resonance conditions which we will assume satisfied in order to simplify our exposition. It follows that the "inward" and "outward" parts of the homoclinic orbit q are respectively identified to the u and v axes.
We can now sketch the construction of a "first return map" in the vicinity of q. For this we choose a transverse "inward" section to q at a point $q(t_1)$, for $t_1 > 0$ large enough so that $q(t_1)$ belongs to the local orbit space. We call Σ^{in} this section. Similarly, we define an "outward" section Σ^{out} at $q(t_2)$ for t_2 sufficiently negative. In the local orbit space coordinates,

$$\Sigma^{in} = \{(\delta, v, w)\} \text{ and } \Sigma^{out} = \{(u, \delta, w)\}$$

where δ is a small positive constant, u and v are ≥ 0. These cross sections are shown in Figure 6. We write coordinates in Σ^{in} and Σ^{out} respectively as (v^{in}, w^{in}) and (u^{out}, w^{out}). Let $\Phi : \Sigma^{in} \to \Sigma^{out}$ be the map defined by the local

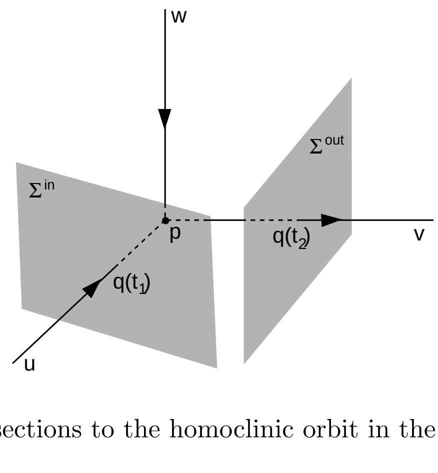

Figure 6: Cross-sections to the homoclinic orbit in the local orbit space.

flow. It is of course not defined in all of Σ^{in}, in particular it is not defined at the point $(0, 0)$ which lies on q. By solving the linear system and eliminating the "time of flight" from the inward section to the outward section, we may nevertheless write

$$\Phi(v^{in}, w^{in}) = \left(\delta(\frac{v^{in}}{\delta})^{\frac{-\lambda_-}{\lambda_+}}, w^{in}(\frac{v^{in}}{\delta})^{\frac{-\lambda_r}{\lambda_+}} \right).$$

and this map is continuous at $(0,0)$.
Now we define the map $\Psi : \Sigma^{out} \to \Sigma^{in}$ defined by the flow "outside" the local region. This is a diffeomorphism which in addition must satisfy the relations $\Psi(0, w^{out}) = (0, w^{in})$ and $\Psi(0,0) = (0,0)$. Therefore Ψ has the form

$$\Psi(u^{out}, w^{out}) = (Au^{out}, Bu^{out} + Cw^{out})$$

where A, B and C are smooth functions of (u^{out}, w^{out}) which in general do not vanish at 0.
We may now define the first return map by composing the two maps:
$\Psi \circ \Phi(v^{in}, w^{in}) = (v_1^{in}, w_1^{in})$, with

$$(v_1^{in}, w_1^{in}) = \left(\delta A(\frac{v^{in}}{\delta})^{\frac{-\lambda_-}{\lambda_+}}, \delta B(\frac{v^{in}}{\delta})^{\frac{-\lambda_-}{\lambda_+}} + Cw^{in}(\frac{v^{in}}{\delta})^{\frac{-\lambda_r}{\lambda_+}} \right).$$

Theorem 4.2 *The map $\Psi \circ \Phi$ is continuous and well-defined in a neighborhood of 0 in Σ^{in}. Under the generic condition $A(0) \neq 0$, this map is a contraction iff $|\lambda_-| > \lambda_+$. Suppose in addition $A(0) > 0$, $\lambda_r < \lambda_-$ and $\lambda_+ = -\lambda_-(1+\mu)$, where μ is a small parameter. Then a family of periodic orbits bifurcate from the homoclinic orbit in the orbit space for either $\mu > 0$ or $\mu < 0$, depending on whether $A(0) < 1$ or $A(0) > 1$.*

The idea of proof of the bifurcation result consists in showing that the "first return map" has a branch of fixed points bifurcating from 0 as μ crosses 0. By setting $r = (\frac{v^{in}}{\delta})^{1/(1+\mu)}$, we are led to solve the system

$$r^{1+\mu} = Ar \quad (19)$$

$$w^{in} = \delta Br + Cw^{in} r^{\frac{\lambda_r}{\lambda_-}}. \quad (20)$$

If $\lambda_r < \lambda_-$, then the second equation can be solved for w^{in} by the implicit function theorem while the resolution of the first equation requires some more work. The details and related bibliography can be found in Chossat and Lauterbach[7]. The periodic orbits which are found in the orbit space correspond to relative periodic orbits in the (z_1, z_2) phase space.

We stress again that the foregoing discussion is only intended for illustration and does not provide a rigorous setting and proofs. However the technical details which are needed to attain full rigour are not related to the orbit space reduction and can be found in the abovementionned bibliography.

References

[1] D. Armbruster, J. Guckenheimer and P. Holmes. *Heteroclinic cycles and modulated waves in systems with* **O(2)** *symmetry*, Physica D **29**, 257-282 (1988).

[2] T.J. Bridges. *$O(2)$ invariant Hamiltonians on $\mathbb{C}^4$ and the (m,n) mode interaction problem for capillary-gravity waves*, Studies Appl. Math. **82**, 93-120 (1990).

[3] P. Chossat. *Forced reflectional symmetry-breaking of an $O(2)$-symmetric homoclinic cycle*, Nonlinearity **6**, 723-731 (1993).

[4] P. Chossat, F. Dias. *The 1:2 resonance with $O(2)$ symmetry and its applications in hydrodynamics*, J. Nonlinear Sci. **5**, 105-129 (1995).

[5] P. Chossat, L. Lauterbach. *Théorème de Hartman-Grobman et réduction à l'espace des orbites*. C.R. Acad. Sci. Paris, Sér. I, 325, 595-600 (1999).

[6] P. Chossat, F. Guyard, R. Lauterbach. *Generalized heteroclinic cycles in sphericaly invariant systems and their perturbations*, J. Nonlinear Sci. **9**, p. 479-524 (1999).

[7] P. Chossat, R. Lauterbach. *Methods in equivariant bifurcation and dynamical systems*, Advanced series in nonlinear dynamics **15**, World Scientific, Singapur (2000).

[8] S.N. Chow, B. Deng, B. Fiedler. *Homoclinic bifurcation at resonant eigenvalues*, J. Dyn. Diff. Equ. **2**, 177-244 (1990).

[9] P. Christodoulides, F. Dias. *Resonant capillary-gravity interfacial waves*, J. Fluid Mech. **265**, 303-343 (1994).

[10] R.H. Cushman, L.M. Bates. *Global Aspects of Classical integrable systems*, Birkhäuser Verlag, Basel (1997).

[11] M. Field. *Equivariant Dynamical Systems*, Trans. Am. Math. Soc. **231** (4), 429-450 (1980).

[12] M. Field. *Equivariant bifurcation theory and symmetry breaking*, J. Dyn. Diff. Equ. **1**, 369-421 (1989).

[13] M. Field. *Symmetry-breaking for compact Lie groups*, Memoirs of the AMS **120** (1996).

[14] M. Golubitsky, D.G. Schaeffer and I. Stewart. *Singularities and Groups in Bifurcation Theory*, Appl. Math. Sci. II, Springer-Verlag, New-York (1988).

[15] J. Guckenheimer and P. Holmes. *Structurally stable heteroclinic cycles*, Math. proc. Cambridge Phil. Soc. **103**, 189-192 (1988).

[16] P. Hartman. *Ordinary Differential Equations*, John Wiley & Sons Corp. (1964).

[17] M. Koenig. *Linearization of vector fields on the orbit space of the action of a compact Lie group*, Math. Proc. Cambridge Phil. Soc. **121**, 401-424 (1996).

[18] M. Krupa. *Bifurcations of relative equilibria*, SIAM J. Math. Anal. **21**, 6, 14531486 (1990).

[19] L. Michel. *Points critiques des fonctions G-invariantes*, C.R. Acad. Sci. Paris, série A-B **272**, A433-A436 (1971).

[20] N. Rott. *A multiple pendulum and its demonstration of nonlinear coupling*, Z. Angew. Math. Phys. **21**, 570-582 (1970).

[21] G. Sartori. *Geometric invariant theory: a model-independent approach to spontaneous symmetry breaking and/or supersymmetry breaking.* Rivista del Nuovo Cimento **14**, 11, 1-120 (1991).

[22] G. Schwarz. *Smooth functions invariant under the action of a compact Lie group.*Topology, **14**, 63-68 (1975).

[23] J. Swift. *Hopf bifurcation with the symmetry of the square*, Nonlinearity **1** (2), 333-377 (1988).