Nonlinear symmetries and normal forms

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Abstract: We provide a unified treatment – including some new results and improvement of old ones – of normal form theory of dynamical systems (or vector fields) for what regards its symmetry properties under nonlinear transformations (Lie point-symmetries). We also discuss the problem of the convergence of the normalizing transformations in the presence of symmetries.

1 Introduction

Normal Form (NF) theory[2, 3, 6, 7, 9, 10, 34] is an old subject in the study of Dynamical Systems (DS), having been created by Poincaré in his Thesis, but it is still one of the most useful and used tools in the study of DS, and a basic building block in the understanding of these.

It is well known that a system in NF can be also characterized by its symmetry properties[3, 5, 10, 22, 28]; a number of recent papers have investigated the consequences of the presence of other symmetries, beside those connected with Poincaré NF. These consequences are twofold: on the one side, they restrict the form of the NF; on the other, they can even guarantee the normalizing transformation to be convergent.

In the present note we provide a unified treatment of several results scattered in the literature, together with some recent results and improvements of old ones. For ease of discussion we will focus on the case where the basic linear operators are semisimple ones; however a substantial part of the theory will go over to the general case (and the Hamiltonian one as well) as we briefly mention below and discuss more in detail in[21].

Although we deal in general with possibly purely formal power series, we will also briefly discuss the problem of the convergence of the normalizing transformations. Also in this context, the presence of symmetry properties plays an important role.

2 Normal Forms

We will consider a analytic DS defined in a neighbourhood of the origin in $\mathbf{R}^n$, and such that the origin is a fixed point of the DS. Equivalently, we can think of studying the Taylor expansion of a DS on a manifold about one of its critical point. We have

$$\dot{x} = f(x) = Ax + F(x) = Ax + \sum_{k \geq 2} F_k(x) \tag{1}$$

where $A = (Df)(0)$ and each F_k homogeneous of degree k. Clearly, to (1) is associated a vector field (VF) $X_f \in \mathcal{V}$, where $\mathcal{V}$ is the space of $n-$dimensional real analytic VF,

$$X_f = f_i(x) \frac{\partial}{\partial x_i} \equiv (f \cdot \nabla) \tag{2}$$

In the following, for ease of language, we will mention "the VF $f \in \mathcal{V}$" to mean "the VF $X_f \in \mathcal{V}$".

A special role is played by the matrix A; we assume, unless otherwise stated, that:

Assumption S. *The matrix $A = (Df)(0)$ is semi-simple, i.e. diagonalizable.*

We can then work in the basis in which A is diagonal.

Definition 1. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the eigenvalues of A; then A is **resonant** if there are non-negative integers $\{q_1, \ldots, q_n\}$ and $r \in [1, n]$ such that

$$(Q \cdot \Lambda) \equiv \sum_{i=1}^{n} q_i \lambda_i = \lambda_r \qquad \text{with} \qquad |Q| \equiv \sum_{i=1}^{n} q_i \geq 2 \tag{3}$$

and the resonance is said to be of the order $|Q|$.

Definition 2. A coordinate transformation of the form

$$x \to \tilde{x} = x + h_k(x) \tag{4}$$

with h_k homogeneous of degree k, $k \geq 2$, is said to be a **Poincaré** (P) **transformation**.

Theorem 1. (Poincaré) *If A is non-resonant, then the DS* (1) *can be reduced to its linear part by a formal series of P transformations; if A does not have resonances of order k or lower, then* (1) *can be reduced to its linear part up to terms of order $O(k+1)$.*

With the definition, related to Definition 1, of **resonant monomials** as those vector-valued monomials with components of the form $x^Q \equiv x_1^{q_1} x_2^{q_2} \cdot \ldots \cdot x_n^{q_n}$

where $Q \equiv (q_1, \ldots, q_n)$ satisfies a relation of the kind of (3), and of **resonant vectors** as those which are a combination of resonant monomials, we have

Theorem 2. (Dulac) *If A is resonant, the DS (1) can be reduced by means of a sequence of formal P transformations to resonant form, namely to the form $\dot{x} = Ax + \sum_{k\geq 2} \widehat{F}_k(x)$, where the $\widehat{F}_k$ are resonant vectors.*

Definition 3. The DS (1) with A semisimple is said to be in **Poincaré-Dulac** (PD) **normal form** if all the nonlinear terms F_k are resonant with A.

Then, Theorems 1 and 2 can be restated[2] as the

Theorem 3. (Poincaré - Dulac). *The DS* (1) *can be brought to NF by means of a series of formal P transformations.*

In general, these transformations will only be formal, i.e. we cannot guarantee the convergence of the series they generate in any neighbourhood - no matter how small - of the origin. Thus, the resulting NF will in general be only a formal series.

Remark 1. In concrete applications, e.g. for numerical computations, one does however consider truncated series, so that the formal or asymptotic series can be as useful as a convergent one. Nevertheless, in some cases it is of interest to have convergent series, e.g. in considering conservation properties. Notice also that when considering truncation of the series, approximate symmetries and approximate constants of motion[19, 20] can be as useful as exact ones.

For any series f we have a VF X_f; the VF corresponding to (formal) series form a Lie algebra under the usual Lie bracket $[\cdot, \cdot]$.

We introduce the following Lie-Poisson bracket $\{\cdot, \cdot\}$ in the space of vector functions

$$\{f,\ g\} = (f \cdot \nabla)g - (g \cdot \nabla)f \tag{5}$$

which satisfies in particular $[X_f,\ X_g] = X_{\{f,\ g\}} = \{f,\ g\} \cdot \nabla$.

Definition 4. The **homological operator** $\mathcal{A}$ (sometimes denoted also by $\mathcal{L}_A$) associated to the matrix A is a linear operator defined as

$$\mathcal{A}(f) = \{Ax,\ f\} \ = \ (Ax) \cdot \nabla f - Af \ . \tag{6}$$

Notice that if A, B are two $n \times n$ matrices and $C = [A, B]$ their commutator, one has from (5) $\{Ax, Bx\} = (BA - AB)x = -Cx$. An easy application of the Jacobi identity and this result get the following

Lemma 1. *If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are respectively the homological operators associated to the matrices A, B and $C = [A, B]$, then $[\mathcal{A},\ \mathcal{B}] = -\mathcal{C}$.*

The relevance of the above definitions is due to the following results.

Lemma 2. *By applying the P transformation $x \to \tilde{x} = x + h_m(x)$ to the DS* (1)*, the terms F_k with $k < m$ remain unchanged, and F_m is changed to $\tilde{F}_m = F_m + \mathcal{A}(h_m)$.*

Remark 2. If assumption S does not hold, i.e. A is not semisimple, then the relevant operator in Lemma 2 above (as well as in other results of this section, and in the definition of resonance) would be the adjoint of the homological operator, $\mathcal{A}^+$. It is also known[22] that, considering the subspaces $\mathcal{V}_{(J)}$ of the homogeneous vector polynomials with components $x^J \equiv x_1^{j_1} x_2^{j_2} \cdot \ldots \cdot x_n^{j_n}$ of degree $|J| = j_1 + \ldots + j_n$, one has $\mathcal{A} : \mathcal{V}_{(J)} \to \mathcal{V}_{(J)}$, and a scalar product (see [22] for the explicit definition) can be introduced in each $\mathcal{V}_{(J)}$. With a suitable choice of the scalar product (the Bargmann one[22]) in $\mathcal{V}_{(J)}$, we have simply $\mathcal{A}^+ = \{A^+x, .\}$.

Lemma 3. *Any term F_k which is in Ran($\mathcal{A}$) can be eliminated by a P transformation with suitable h_k.*

Lemma 2 follows from explicit computation, and Lemma 3 is then an obvious corollary. Indeed it is in this way that the PD theorem is proved; in fact, one can prove, using also Assumption S (or we would have $\mathcal{A}^+$ instead of $\mathcal{A}$):

Lemma 4. *The term F_k is resonant with A if $F_k \in \mathrm{Ker}(\mathcal{A})$.*

Remark 3. In the Lemmas 2-4 above, we have used Assumption S to guarantee that Ran($\mathcal{A}$) is a complementary subspace of Ker($\mathcal{A}$). If in particular the matrix A is normal, i.e. $[A, A^+] = 0$ (notice however that any semisimple A can always be transformed into a normal A by means of a linear transformation, and any normal A is semisimple), then Ker($\mathcal{A}$) is precisely the orthogonal complement of Ran($\mathcal{A}$): see Remark 2 and [22, 17] for a precise discussion.

Therefore, we have equivalently to Definition 3 the

Definition 5. The DS (1) is in (PD) NF if it is satisfied $f \in \mathrm{Ker}(\mathcal{A})$ or equivalently $\{Ax, f\} \equiv \mathcal{A}(f) = 0$.

3 Symmetries

It is now time to consider the symmetry properties of (2.1).

Definition 1. The series $g(x) = Bx + G(x)$, where $B = (Dg)(0)$, is the infinitesimal generator of a symmetry of (2.1) if $[X_f, X_g] = 0$, i.e. equivalently if $\{f(x),\ g(x)\} = 0$.

These symmetries are precisely the (time-independent) Lie point symmetries of the DS[23, 32, 33].

Remark 1. Any DS (2.1) admits the symmetries $g(x) = cf(x)$, $c \in \mathbf{R}$; these will be referred to as trivial symmetries.

We can restate the definition of NF, for A satisfying Assumption S, as

Definition 2. The DS (2.1) is in (PD) NF if the linear function Ax is a symmetry of f.

Let us now assume that the DS (2.1) admits a nontrivial symmetry; i.e. there is a $g \neq cf$ such that $[X_f, X_g] = 0$. Clearly, this geometrical relation does not depend on the coordinate representation of X_f and X_g, i.e. is conserved upon passing to coordinates in which (2.1) is in NF.

We denote by $\mathcal{G}_f$ the algebra of symmetries of f:

$$\mathcal{G}_f \equiv \big\{g : \ \ [X_f,\ X_g] = 0\big\} \equiv \big\{g : \ \{f,\ g\} = 0\big\} \ . \tag{1}$$

We will also denote by $\mathcal{D}_n$ the algebra of (polynomial) VFs on $\mathbf{R}^n$, and by $\mathcal{M} \subset \mathcal{D}_n$ the algebra of linear VF on $\mathbf{R}^n$, i.e. of VF of the form $f(x) = Mx$ with M any $n \times n$ matrix, introducing also the shorthand notation X_M for the VF X_{Mx}; and by $\mathcal{M}_f$ the subalgebra of **linear** symmetries of f:

$$\mathcal{M}_f \equiv \big\{Bx : \ [X_B,\ X_f] = 0\big\} \equiv \big\{Bx : \ \{Bx,\ f\} = 0\big\} \subseteq \mathcal{G}_f \ . \tag{2}$$

Clearly, $\mathcal{G}_f$ is nothing else that the centralizer of f in the algebra $\mathcal{D}_n$, and $\mathcal{M}_f$ is just $\mathcal{M}_f = \mathcal{G}_f \cap \mathcal{M}$. We will usually write $B \in \mathcal{M}_f$ instead of $Bx \in \mathcal{M}_f$.

It should be stressed that $\mathcal{G}_f$ has not only the structure of algebra, but it is also a **module**. Indeed, let $\mathcal{I}_f$ be the set of constants of motion, or first integrals, for X_f; i.e. $\mathcal{I}_f \equiv \{\rho : \ \mathbf{R}^n \to \mathbf{R} : \quad X_f(\rho) = 0\}$. It is immediate to check that $\mathcal{I}_f$ is an algebra, and by trivial computations we have that if $g_1, \ g_2 \in \mathcal{G}_f$ and $\mu_1, \mu_2 \in \mathcal{I}_f$, then $(\mu_1 g_1 + \mu_2 g_2) \in \mathcal{G}_f$. This means that (see also [35]):

Lemma 1. *The set $\mathcal{G}_f$ is a finitely generated module over $\mathcal{I}_f$.*

Remark 2. Notice that if f admits nontrivial constants of motion, then $\mathcal{G}_f$ will be infinite dimensional as an algebra, even if it is finite dimensional as a module.

Lemma 2. *The set $\mathcal{I}_f$ is invariant under $\mathcal{G}_f$: $X_g : \mathcal{I}_f \to \mathcal{I}_f \ \forall g \in \mathcal{G}_f$.*

Indeed, if $\mu \in \mathcal{I}_f$, $X_g X_f(\mu) = 0$; this also means that $X_f\big(X_g(\mu)\big) = 0$, hence $X_g(\mathcal{I}_f) \subseteq \mathcal{I}_f$.

Let us now assume for the symmetries $g = Bx + G(x)$ a property analogous to Assumption S for the VF f, namely:

Assumption S$'$. *The matrix $B = (Dg)(0)$ is semisimple.*

When a (sequence of) P transformations is performed in order to bring f to normal form, a different series g is, in general, not brought to NF. However, we have[17]:

Theorem 1. *Let $f(x)$ and $g(x)$ be such that $\{f, \ g\} = 0$; then, the formal P transformation changing f into NF, $f \to \widehat{f} \in \mathrm{Ker}(\mathcal{A})$, also changes g into $\widetilde{g}$ such that $\widetilde{g} \in \mathrm{Ker}(\mathcal{A})$; and there is a formal P transformation bringing both f and g to Joint Normal Form (JNF), i.e.*

$$f \to \widehat{f} \qquad g \to \widehat{g} \qquad \text{where} \qquad \widehat{f}, \ \widehat{g} \in \mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B}) \tag{3}$$

with $\mathcal{A}$, $\mathcal{B}$ the homological operators associated to $A = Df(0)$, $B = Dg(0)$.

Proof (a sketch; for details, see [17]). Let f be put in NF: $f \to \widehat{f} = Ax + \widehat{F}$; under this transformation, the symmetry g will be transformed into a new form, $g \to \widetilde{g} = Bx + \widetilde{G}$ (not necessarily normal; we reserve the notation $\widehat{\cdot}$ to NF only). The commutation property $\{f, g\} = 0$ is clearly independent of the coordinates, then $\{\widehat{f}, \widetilde{g}\} = 0$ holds too. Let us now expand $\widehat{f}$ and $\widetilde{g}$ in formal power series as $\widehat{f} = Ax + \sum_{j \geq 2} \widehat{F}_j$, $g = Bx + \sum_{j \geq 2} \widetilde{G}_j$, where $\widehat{F}_j$ and $\widetilde{G}_j$ are homogeneous polynomials of degree j. Looking at the first-order term in the series expansion, one immediately obtains from $\{\widehat{f}, \widetilde{g}\} = 0$ that $[A, B] = 0$; then, proceeding at order 2, one gets $\{Ax, \widetilde{G}_2\} = \{Bx, \widehat{F}_2\}$. Applying to this equality the operator $\mathcal{A}$, one has $\mathcal{A}^2(\widetilde{G}_2) = 0$, which gives, thanks to the semisimplicity of the matrix A, $\mathcal{A}(\widehat{G}_2) = 0$. At the order $k > 2$, one has $\{Ax, \widetilde{G}_k\} - \{Bx, \widehat{F}_k\} = \sum_{j=2}^{k-1} \{\widehat{F}_j, \widetilde{G}_{k-j+1}\} \equiv \psi_k$. On the other hand, an immediate application of the Jacobi identity shows that if $\widehat{F}_j, \ \widetilde{G}_i \in \mathrm{Ker}(\mathcal{A})$ the same is true for $\{\widehat{F}_j, \widetilde{G}_i\}$. This allows us to proceed inductively to obtain $\widehat{F}, \ \widetilde{G} \in \mathrm{Ker}(\mathcal{A})$, and so $\widehat{f} \ , \widetilde{g} \in \mathrm{Ker}(\mathcal{A})$. But the series of transformations which normalize f are actually determined up to elements in $\mathrm{Ker}(\mathcal{A})$, therefore, being $\widetilde{G} \in \mathrm{Ker}(\mathcal{A})$ and $\mathcal{B} : \mathrm{Ker}(\mathcal{A}) \to \mathrm{Ker}(\mathcal{A})$, one can still apply changes of coordinates to also put $\widetilde{g}$ in NF with respect to B without changing the normality of $\widehat{f}$. The result $\widehat{f} \in \mathrm{Ker}(\mathcal{B})$ automatically follows using the same argument as before to show $\widetilde{g} \in \mathrm{Ker}(\mathcal{A})$, and, in conclusion, the JNF (3) is obtained. $\triangle$

Remark 3. Clearly, the role of f and g in Theorem 1 can be reversed. This fact will be useful in the problem of linearizing a DS, see below.

The result in Theorem 1 can be immediately extended if we consider not only two VF X_f, X_g, but an abelian $(d+1)-$dimensional algebra $\mathcal{G}$:

Theorem 2. *Let f and $\{g_1, \ldots, g_d\}$ form a $(d+1)$-dimensional abelian Lie algebra $\mathcal{G}$ under $\{\cdot,\cdot\}$. Then, by a series of P transformations, $\{f = g_0,\ g_1,\ \ldots,\ g_d\}$ can be put in Joint NF; i.e. $g_b \to \widehat{g}_b \in \bigcap_{a=0}^{d} \mathrm{Ker}(\mathcal{B}_a)$ (for $b = 0, 1, \ldots, d$), where $\mathcal{B}_a$ are the homological operators associated to the matrices $B_a = (Dg_a)(0)$ and $A \equiv B_0$.*

It is worth considering in further detail the relation between $\mathcal{G}_f$ and $\mathcal{M}_f$ defined above. It is often useful to consider G_h, $\mathcal{M}_h$, $\mathcal{I}_h$ for linear VF $h(x) = Ax$, where A is a matrix; in this case we will simply write $\mathcal{G}_A$, $\mathcal{M}_A$, $\mathcal{I}_A$ (similarly to what we do for VF).

Let us restate for convenience the following result, which actually we have already shown and used (in the proof of Theorem 1).

Lemma 3. *If $g = Bx + G(x)$ is a symmetry of $f = Ax + F(x)$ then $[A,\ B] = 0$; i.e. $[X_f,\ X_g] = 0 \Rightarrow [X_A,\ X_B] = 0$.*

Corollary 1. *In particular, considering $g(x) = Bx$, this ensures $\mathcal{M}_f \subseteq \mathcal{M}_A$ and $\mathcal{M}_f \subseteq \mathcal{M}_F$.*

Another important consequence is the following Corollary 2[28]; Corollary 3 follows from Theorems 1-2 and Lemma 4 is proved in [35].

Corollary 2. *If the DS $f = Ax + F$ admits a linear symmetry $g = Bx$, then there is a NF $\widehat{f}$ which admits the same symmetry $g = Bx$, and this NF actually satisfies $\widehat{f} \in \mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B})$.*

Proof. From $\{Bx, f\} = 0$ and $[A, B] = 0$, one deduces that $f \in \mathrm{Ker}(\mathcal{B})$ and $\mathcal{A} : \mathrm{Ker}(\mathcal{B}) \to \mathrm{Ker}(\mathcal{B})$. This ensures that one can put f in NF with respect to A, $f \to \widehat{f} \in \mathrm{Ker}(\mathcal{A})$ remaining in $\mathrm{Ker}(\mathcal{B})$, i.e. $\{Bx, \widehat{f}\} = 0$ and $\widehat{f} \in \mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B})$. $\triangle$

Corollary 3. *If $g = Bx + G \in \mathcal{G}_f$ then, once in JNF, $\widehat{f} \in \mathrm{Ker}(\mathcal{B})$, or $B \in \mathcal{M}_{\widehat{f}}$, i.e. Bx is a linear symmetry of $\widehat{f}$. If $\mathcal{G}_f^*$ is a maximal abelian subalgebra of $\mathcal{G}_f$, spanned by $g_a = B_a x + G_a(x)$ (with $g_0 \equiv f$), then, once put in JNF, $B_a \in \mathcal{M}_f$.*

Lemma 4. [35] *If $f = Ax + F$ is in NF, i.e. $F \in \mathrm{Ker}(\mathcal{A})$, then any constant of motion $\mu(x)$ of $\dot{x} = f$ is also constant of motion of the linearized VF Ax; that is: if $\mu \in \mathcal{I}_f$ then $\mu \in \mathcal{I}_A$, or $\mathcal{I}_f \subseteq \mathcal{I}_A$.*

An application of the JNF deals with the case of *linearizable* DS. Clearly, if $\mathrm{Ker}(\mathcal{A}) = \{0\}$, the DS is linearizable by means of a formal P transformation. But, whatever the matrix A, the linear VF $X_A = Ax \cdot \nabla$ does clearly commute with the VF S given by $S = \sum_i x_i \partial/\partial x_i = (Ix \cdot \nabla)$, which generates the dilations in R^n; it is easy to see, conversely, that the only VF commuting with S are the linear ones. On the other hand, it is also clear that the identity does not admit resonances. Then we can say[4, 26]:

Corollary 4. *A VF f $\big($or a DS $\dot{x} = f(x)\big)$ can be linearized* iff *it admits a (possibly formal) symmetry $g = Bx + G$ such that $B = (Dg)(0) = I$.*

Remark 4. If the VF f is $2-$dimensional and $A = (Df)(0)$ has different eigenvalues, and it admits a symmetry $g = Bx + G$ with B not proportional to A, then it can be linearized. Indeed, it is sufficient to consider the linear combination $af + bg$ giving another symmetry such that $aA + bB = I$.

More in general, we can also say:

Corollary 5. *The DS $\dot{x} = f = Ax + F$ can be linearized if it admits a (possibly formal) symmetry $g = Bx + G$ such that* $\mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B}) = \{0\}$ *(in particular, if B does not admit resonances, i.e.* $\mathrm{Ker}(\mathcal{B}) = \{0\}$*).*

Indeed the JNF of the DS is then $\dot{x} = Ax$, and that of the symmetry reduces to its linear part. In this respect we notice that $\mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B})$ is spanned by vectors $|\mu, s>$ having only the s component different from zero, and this equal to $x_1^{\mu_1} ... x_n^{\mu_n}$, where the monomials x^μ are resonant with both A and B; if λ_i are the eigenvalues of A and β_i those of B, this means that $(\lambda_i \cdot \mu_i) = \lambda_s$ and $(\beta_i \cdot \mu_i) = \beta_s$.

This result can be easily extended not only to the case of more than one symmetry, as an obvious consequence of Theorem 2, but also to the non-semisimple case[18]. Other comments on this point will be given later on when the convergence of the normalizing transformations will be discussed. Another result, related to the properties of linear DS, is the following[18]:

Theorem 3. *If a DS in $\mathbf{R}^n$ can be linearized, then it admits n independent* commuting *symmetries, which can be simultaneously taken into linear form by a coordinate transformation. If, in particular, the system has a diagonalizable A with real eigenvalues, then - once it is linearized and A is diagonal - the dilations $\zeta_i = x_i \partial_i$ (no sum over i) along each direction i, are n linear commuting symmetries for the system. Conversely, if there is a coordinate system where the DS admits n independent linear commuting symmetries $X_i = B_i x$ such that all B_i are semisimple, then the DS can be linearized.*

Proof. In the coordinates where the DS is linear, it is easy to construct n linear commuting symmetries $B_i x \partial_x$, simply choosing n independent matrices B_i commuting among themselves and with A (if the matrices $I, A, A^2, \ldots, A^{n-1}$ are linearly independent, they immediately provide the matrices B_i ; even if this is not the case, the existence of n matrices B_i with the required properties is easily verified if A is put in Jordan form). The existence of the n independent scalings ζ_i in the case of diagonal A is obvious. Conversely, given n semisimple commuting matrices B_i, they can be simultaneously diagonalized: $B_i \to \mathrm{diag}\ (\beta_1^{(i)}, \ldots, \beta_n^{(i)})$; now, with respect to the basis spanned by the n (independent) vectors $\beta^{(i)} \equiv (\beta_1^{(i)}, \ldots, \beta_n^{(i)})$, the symmetries $\sigma_i = B_i x \partial_x$ become $\sigma_i \to \zeta_i = x_i \partial_i$ (no sum over i), i.e. the independent dilations along the directions $\beta^{(i)}$. A DS admitting such n symmetries is necessarily linear. $\triangle$

4 On linear and nonlinear symmetries

The relation between linear and nonlinear symmetries, especially the "quasi linear" symmetries to be considered in a moment, does sometimes lead to some confusion; in this section we would like to clarify some points in this respect, due also to its relevance for the problem of convergence.

It should be stressed that when we apply a series of P transformations to bring f into NF, by applying the same series of P transformations to a different VF g, in general not only there is no reason this is also brought to NF (as already mentioned), but even if g is of low order, under the series of P transformations it will generate terms of all orders. However if g is a symmetry of f, the situation is nongeneric, and we have some interesting results.

The first of these is that, if f admits a linear symmetry $g(x) = Bx$, then not only Bx is still a symmetry of $\widehat{f}$ in NF (Corollary 3.2), but also it is possible to choose the series of P transformations taking f into NF in such a way that $g = Bx$ is not changed (see also [5, 22, 28]). Moreover, in this case the normalising transformation itself can be chosen to admit $g(x) = Bx$ as a symmetry[22, 28]:

Theorem 1. *If $f(x)$ commutes with $g(x) = Bx$, then it is possible to find a series of P transformations with generators $h_m(x)$, $m = 2, \ldots$, such that f is taken to PD normal form and g is left unchanged. Moreover, these h_m do commute with g, i.e. $\{Bx, h_m(x)\} = 0 \ \forall m$.*

The proof of this relies on a similar result obtained by Ruelle for the Center Manifold mapping[28] and we omit it.

We will denote by $\mathcal{Q}_f$ the set of nonlinear symmetries which are *"quasi linear"*[29], i.e. are linear combinations of terms of the form

$$\{g \in \mathcal{G}_f : \ g(x) = \mu(x) \ Bx, \ \mu(x) \in \mathcal{I}_f, \ B \in \mathcal{M}_f\} \ . \tag{1}$$

We write then $\mathcal{Q}_f = \mathcal{I}_f \otimes \mathcal{M}_f$ (this will mean the module over $\mathcal{I}_f$ generated by $\mathcal{M}_f$, and we use the same notation later on for $\mathcal{G}_f$, $\mathcal{G}_A$); obviously, $\mathcal{Q}_f \subseteq \mathcal{G}_f$.

Notice that $\mathcal{Q}_f$ is, clearly, a module over the algebra $\mathcal{I}_f$ of constants of motion of f; we stress in particular that it is $\mathcal{I}_f$ which intervenes in this, and *not* $\mathcal{I}_A$; also, it is $\mathcal{M}_f$ which intervenes in this, and *not* $\mathcal{M}_A$.

Remark 1. It should be stressed – as sometimes some confusion is possible – that in general $\mathcal{Q}_f \neq \mathcal{G}_f$, i.e. not all the $g(x) \in \mathcal{G}_f$ are quasi-linear[17, 18]. Similarly, in general $\mathcal{I}_f \neq \mathcal{I}_A$, even if f is in NF (see Lemma 3.4 for this case). Also, if $[B, \ A] = 0$ in general $\mu(x) \ Bx$ does not belong to $\mathcal{G}_f$, even for $\mu \in \mathcal{I}_f$ (unless $B \in \mathcal{M}_f$ as well), nor to $\mathcal{G}_A$ (unless $\mu \in \mathcal{I}_A$); see however below in this section.

Thus, one should avoid confusions between linear symmetries of the DS and symmetries of its linearisation, which do not extend in general to symmetries of the full system.

Example 1. To stress the points made in the above remark, consider the 2-dimensional case

$$\dot{x} = \alpha(\rho) J x \equiv f(x) \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ; \ \ \rho = x_1^2 + x_2^2 \ . \tag{2}$$

Then, if $\alpha(0) \neq 0$ and $\alpha(\rho) \neq c$, we have $\mathcal{G}_f = \{\mu(\rho)Jx\}$ with μ arbitrary, i.e. $\mathcal{G}_f \equiv \mathcal{Q}_f$; and $\mathcal{M}_f = J$, so that in this case $\mathcal{G}_f = \mathcal{I}_f \otimes \mathcal{M}_f$, but $\mathcal{M}_f \neq \mathcal{M}_A$; indeed, e.g., the identity matrix I belongs to $\mathcal{M}_A$, but not to $\mathcal{M}_f$.

Example 2. Another example is obtained considering the following modification of the case above: $\dot{x} = \alpha(\rho)Jx + \beta(\rho)x \equiv f(x)$ (notice that f is again in NF); one has that ρ is a constant of motion of the linear part $\dot{x} = Jx$, but not of the full problem; so $\mathcal{I}_f \neq \mathcal{I}_A$.

It should be stressed that, when it happens that $\mathcal{G}_f = \mathcal{Q}_f$, or $\mathcal{I}_f = \mathcal{I}_A$, or $\mathcal{G}_f = \mathcal{G}_A$, then the analysis of symmetry and symmetry-related properties of the DS (2.1) becomes easier, and stronger results hold. We do briefly comment on some such cases.

Remark 2. If A is nonresonant, then f can be reduced to its linear part Ax; in this case, as the algebra $\mathcal{G}_f$ is the same before and after the normalizing transformation (although with different coordinate representations), $\mathcal{G}_f = \mathcal{G}_A$.

Remark 3. Consider the case where f can be transformed by P transformations into the form (which is not necessarily the NF)

$$f(x) = \big(1 + \mu(x)\big) \; Ax \tag{3}$$

where $\mu(x)$ is a real-valued scalar function (see below for the relevance of this case). It is clear that (3) implies $\mathcal{I}_f = \mathcal{I}_A$; it is also easy to get convinced that $\mathcal{G}_f \neq \mathcal{G}_A$, as the identity matrix belongs to $\mathcal{M}_A \subseteq \mathcal{G}_A$ but does not belong [unless $\mu(x) \equiv 0$] to $\mathcal{M}_f \subseteq \mathcal{G}_f$.

No definite relation exists, in general, between $\mathcal{G}_f$ and $\mathcal{G}_A$; but if the DS is in NF, one has the following result:

Lemma 1. *If $f = Ax + F$ is in NF, then any symmetry $g \in \mathcal{G}_f$ is also symmetry for the linear part $\dot{x} = Ax$, i.e. $\mathcal{G}_f \subseteq \mathcal{G}_A$.*

This result can be proved along similar lines as for Theorems 3.1-2, looking at lowest order terms of series expansions[17, 35], and can be a considerable help in the symmetry analysis, as $\mathcal{G}_A$ can be determined in a relatively simple way, solving the system of quasilinear non-homogeneous first-order PDEs $\{Ax, g\} = 0$, which is written explicitly as $(A_{ij}x_j)\,(\partial g_k / \partial x_i) = A_{kj} g_j$. One obtains indeed from this, introducing the set $\mathcal{I}^*_A$ of the meromorphic (i.e. fractions of formal power series) of first integrals of $\dot{x} = Ax$ [18, 22, 35]:

Lemma 2. *$\mathcal{G}_A$ is the set of all formal power series in $\mathcal{I}^*_A \otimes \mathcal{M}_A$.*

Corollary 1. *If f is in NF, then $\mathcal{M}_f \subseteq \mathcal{G}_f \subseteq \mathcal{G}_A$ and $\mathcal{M}_f \subseteq \mathcal{M}_A \subseteq \mathcal{G}_A$.*

Being $\mathcal{G}_A \equiv \mathrm{Ker}(\mathcal{A})$, the above Lemma 2 also provides the most general VF in NF with respect to A, i.e.:

Lemma 3. *The resonant terms $\widehat{F} \in \mathrm{Ker}(\mathcal{A})$ are power series of the form $\widehat{F}(x) = K\big(\rho(x)\big) \cdot x$, where $K \in \mathcal{M}_A$ and $\rho \in \mathcal{I}^*_A$; that is, K is a matrix which commutes with A, and its entries are functions of the time-independent constants of motion $\rho(x)$ for the linear problem $\dot{x} = Ax$. Notice, in particular, that the set $\mathcal{G}_A$ of these VF is naturally a Lie algebra.*

It should be stressed that the constants of motion $\rho(x)$ which appear in the above Lemmas are expressed not only as (formal) power series but also possibly in the form of rational functions (see [35] for a complete description of this point). Notice also that the results given for systems in NF can be no longer true if the DS is not in NF, as the following example shows.

Example 3. Consider the following 2-dimensional problem, which is *not* in NF: $\dot{x}_1 = -x_1 x_2$, $\dot{x}_2 = x_2$. It is easily seen that $\mu = x_1 \exp(x_2)$ is a constant of motion of the the DS, but not of the linearized problem: $\mu \in \mathcal{I}_f$, but $\mu \notin \mathcal{I}_A$; similarly, $X_g = x_1^2 \exp(x_2)(\partial/\partial x_1)$ is a symmetry for the DS. i.e. $g \in \mathcal{G}_f$, but $g \notin \mathcal{G}_A$.

Remark 4. Recently, Kodama[29] considered the problem of determining $\mathcal{G}_A$ from a more algebraic standpoint. In the same work, Kodama also observed that $\mathcal{G}_A$, considered as an algebra, is not only infinite dimensional, but has the natural structure of a graded Virasoro algebra.

5 The non-semisimple case

As mentioned above, most of the results considered here for A semisimple also hold – sometimes in a weaker form – if Assumptions S and S' are dropped, although in this case the notation required becomes more involved. In this section we briefly discuss the non-semisimple case, referring to [21] for a complete discussion and for proofs of our statements.

It is a well known result (see e.g. [27]) that a generic matrix A is uniquely decomposed as $A = A_s + A_n$, where A_s is semisimple (i.e. diagonalizable), A_n is nilpotent, and these commute, $[A_s, A_n] = 0$. Also, if $[A, B] = 0$ and $A = A_s + A_n$, $B = B_s + B_n$, then $[A_s, B] = [A, B_s] = 0$ (the last statement easily follows by writing either one of the matrices A, B in Jordan form and imposing commutation in generalised eigenspaces).

In the case of matrices with a nilpotent part, the definition of resonant terms will be slightly modified, introducing the homological operators $\mathcal{A}_s$ and $\mathcal{A}^+$ associated to A_s and A^+ (see also Remark 2.2): $\mathcal{A}_s = \{A_s x, \cdot\}$, $\mathcal{A}^+ = \{A^+ x, \cdot\}$. Later on we will use $\mathcal{B}, \mathcal{B}_s, \mathcal{B}^+$ defined similarly. We assume we have chosen the Bargmann scalar product[22, 28] in $\mathcal{V}_k$.

Definition 1. $f(x) \in \mathcal{V}$ is resonant with the matrix A if $f(x) \in \mathrm{Ker}(\mathcal{A}^+)$, i.e. $\{A^+ x, f(x)\} = 0$.

Lemma 1. *The terms $\widehat{F}(x) \in \mathrm{Ker}(\mathcal{A}^+)$ are power series of the form $K\big(\rho(x)\big) \cdot x$, where K is a matrix which commutes with A^+, and its entries are functions of the time-independent constants of motion $\rho(x) \in \mathcal{I}^*_{A^+}$ for the linear problem $\dot{x} = A^+ x$.*

From explicit computations of the time-independent constants of motion of linear problems $\dot{x} = Ax$ with non-semisimple A, we can easily show[18] (cf. also [2, 31], where this is viewed in the context of the theory of replicas), using the above Lemma 1 and the analyticity of the elements in $\mathrm{Ker}(\mathcal{A})$ and $\mathrm{Ker}(\mathcal{A}^+)$, that $\mathrm{Ker}(\mathcal{A}) \subseteq \mathrm{Ker}(\mathcal{A}_s)$ and $\mathrm{Ker}(\mathcal{A}^+) \subseteq \mathrm{Ker}(\mathcal{A}_s) = \mathrm{Ker}(\mathcal{A}_s^+)$. Here, we can assume (see Remarks 2.2-3) that the semisimple part A_s of the matrix A is a normal matrix, $[A_s, A_s^+] = 0$.

Remark 1. The Poincaré-Dulac theorem, reformulated in the non-semisimple case, states that, given any VF $f = Ax + F$, the nonlinear part can be transformed by means of a formal P transformation, into NF with respect to $\mathcal{A}^+$, i.e. $F \to \widehat{F} \in \mathrm{Ker}(\mathcal{A}^+)$. Notice that, in this case, the resonance condition $\mathcal{A}^+(\widehat{F}) = 0$ cannot be extended to the linear term: $Ax \notin \mathrm{Ker}(\mathcal{A}^+)$. However, $Ax \in \mathrm{Ker}(\mathcal{A}_s)$, and therefore one has $\widehat{f} \in \mathrm{Ker}(\mathcal{A}_s)$.

The counterpart to Theorem 3.1 for the non semisimple case is given by the following Theorem[1, 18].

Theorem 1. *Let $f(x) = Ax + F(x)$, $g(x) = Bx + G(x)$ satisfy $[X_f, X_g] = 0$, and let the semisimple parts A_s and B_s of A and B be normal matrices; then, by means of P transformations, f, g can be formally taken into either one of the following Joint Normal Forms:*
i) *the "A-JNF": $F \to \widehat{F} \in \mathrm{Ker}(\mathcal{A}^+) \cap \mathrm{Ker}(\mathcal{B}_s)$, $g \to \widehat{g} \in \mathrm{Ker}(\mathcal{A}_s) \cap \mathrm{Ker}(\mathcal{B}_s)$;*
ii) *the "B-JNF": $f \to \widehat{f} \in \mathrm{Ker}(\mathcal{A}_s) \cap \mathrm{Ker}(\mathcal{B}_s)$, $G \to \widehat{G} \in \mathrm{Ker}(\mathcal{A}_s) \cap \mathrm{Ker}(\mathcal{B}^+)$;*
iii) *the "Semisimple JNF" (SJNF), actually included in both cases:*
$f,\ g \to \widehat{f},\ \widehat{g} \in \mathrm{Ker}(\mathcal{A}_s) \cap \mathrm{Ker}(\mathcal{B}_s)$.

The proof[21] is similar to that of Theorem 3.1. This result can be easily extended, as in Theorem 3.2, to the case of any abelian algebra[21], with a similar statement (which we omit for the sake of brevity).

For *non-commutative* algebras, it is in general hard (or impossible) to obtain similar results. An important exception is given by *nilpotent* algebras[1, 31], for which indeed the following theorem holds[1, 18].

Theorem 2. *Let $\mathcal{G}$ a Lie algebra and $\mathcal{G}^*$ a nilpotent subalgebra spanned by $g_a = B_a x + G_a$ $(a = 1, \ldots, d^*)$. Then the same JNF as in Theorem 1 can be obtained by formal P transformations.*

Remark 2. Notice that in the above Theorems we have explicitly assumed that the semisimple parts of the involved matrices are written as normal matrices: in fact, this can be obtained by means of a linear transformation, not only in the case of a single matrix - as already noted - but also for a nilpotent (or in particular, abelian) algebra of matrices.

Other Partial JNF can be obtained in the case of *solvable* algebras. Some general results[21] are:

Theorem 3. *Assume that the DS $\dot{x} = f(x)$ admits a $d-$dimensional solvable algebra of symmetries $\mathcal{G} = \{g_1, \ldots, g_d\}$: $\{f, g_i\} = 0$, $g_i = B_i x + G_i$, $(i = 1, \ldots, d)$. Then, by means of a formal P transformation, the following NF can be obtained $f \to \widehat{f} \in \left(\bigcap_{i=1}^{d} \mathrm{Ker}(\mathcal{B}_{i,s})\right) \cap \mathrm{Ker}(\mathcal{A}^+)$, $g_i \to \widetilde{g}_i \in \mathrm{Ker}(\mathcal{A}_s)$*

Theorem 4. *Let $\mathcal{G}$ be a $(d_a + d_b)-$dimensional algebra, spanned by $f_a = A_a x + F_a$ $(a = 1, \ldots, d_a)$ and $g_b = B_b x + G_b$ $(b = 1, \ldots, d_b)$, obeying $\{f_a, f_{a'}\} = \{g_b, g_{b'}\} = 0$; $\{f_a, g_b\} = c_{ab} g_b$ (*no sum over *b), where c_{ab} are arbitrary constants. Then, for each fixed b, there is a formal P transformation which takes g_b into the NF $G_b \to \widehat{G}_b \in \mathrm{Ker}(\mathcal{B}_b^+)$ and all the F_a into $F_a \to \widetilde{F}_a \in \mathrm{Ker}(\mathcal{B}_b)$.*

Remark 3. In the simple case of a two dimensional solvable algebra (relevant when we have an *orbital symmetry[36]*), this means that if $f = Ax + F$, $g = Bx + G$ satisfy $\{f, g\} = g$, then there is a formal P transformation such that $F \to \widetilde{F} \in \mathrm{Ker}(\mathcal{B})$ and $G \to \widehat{G} \in \mathrm{Ker}(\mathcal{B}^+)$

Some other "Partial" JNF can obtained under special assumptions; results in this direction are discussed in [21], covering also cases of special interest in the Hamiltonian case; again, we omit to discuss these for lack of space.

6 Convergence of the normalizing transformation.

As already remarked, the normalizing transformation (NT) – that is, the sequence of P transformations needed to bring the given DS $f(x)$ or VF X_f into NF – is in general only formal, as we are not sure the radius of convergence of the series is greater than zero. It would definitely be of interest to characterise convergence or divergence of the NT in terms of the DS, and we shall see that the symmetry properties of the system can play some role in this problem. In the remaining sections, the matrix $A = (Df)(0)$ will be assumed to be semisimple.

A sufficient condition which ensures convergence of the NT is given by the following well known Poincaré criterion (see [2, 3] for more details).

Theorem 1. (Poincaré) *If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $A = (Df)(0)$ of the analytic VF f belong to the Poincaré domain, i.e. if the convex hull of the points $\lambda_1, \ldots, \lambda_n$ in the complex plane does not contain zero, then the NT is convergent and the NF is analytic.*

There are also two basic and very general conditions, given by Bruno, and called Condition A and Condition ω (see [6, 7] for any detail), which ensure that a vector function $f = Ax + F$ can be put in NF by a convergent transformation[6, 7]. For convenience, we state here Condition A in its simplest (and quite restrictive) form: to this end, let us assume for a moment that there is a straight line through the origin in the complex plane which contains all the eigenvalues λ_i of A (i.e. $\lambda_i = c_i \lambda_0$ for some $\lambda_0 \in \mathbf{C}$ and with $c_i \in \mathbf{R}$; if all the c_i have the same sign convergence is also ensured by theorem 1 above); notice for A real this imply the λ_i are either all real or all pure imaginary. Then the condition reads:

Condition A: *A vector field $\widehat{f}$ in NF is said to satisfy Condition A if $\widehat{f}$ has the form $\widehat{f} = Ax + \alpha(x) Ax$ and $\alpha(x)$ is some scalar-valued power series [with $\alpha(0) = 0$].*

Actually, depending on the position of the eigenvalues of A in the complex plane, there are different versions of Condition A, and the above statement may be modified accordingly[6]. We notice, however, that in many applications, and in particular when linear NF will be ultimately concerned, the above formulation of Condition A is sufficient. The other condition is:

Condition ω: *Let $\omega_k = \min |(Q, \Lambda) - \lambda_j|$ for all $j = 1, \ldots, n$ and all $n-$uple of nonnegative integers q_i such that $1 < \sum_{i=1}^n q_i < 2^k$ and $(Q, \Lambda) = \sum_i q_i \lambda_i \neq \lambda_j$: then $\sum_{k=1}^\infty 2^{-k} \ln \left(\omega_k^{-1}\right) < \infty$.*

This is a very weak condition, devised to control the appearance of small divisors in the series of NT, and generalizes the Siegel-type condition that $\exists \varepsilon, \nu > 0$ such that $|(Q, \Lambda) - \lambda_j| > \varepsilon \, [\sum_{i=1}^n |q_i|]^{-\nu}$, or the simpler condition $|(q, \lambda) - \lambda_j| > \varepsilon > 0$, for all $n-$uple q_i such that $(Q, \Lambda) \neq \lambda_j$ (see [6, 7]). With these two conditions, one has then[6, 7]:

Theorem 2. (Bruno) *If $A = (Df)(0)$ satisfies Condition ω, and f can be transformed, via P transformations, to a NF $\widehat{f}$ which satisfies Condition A, then there is a convergent NT for f .*

Here and in the following, "convergence" stands for "convergence in some open neighbourhood of the stationary point $x_0 = 0$". We will also assume that A always satisfies Condition ω; i.e. we take for granted from now on that

Assumption B. *The matrix $A = (Df)(0)$ satisfies Condition ω.*

7 Normalizing transformations in the presence of symmetries.

The presence of some symmetry g of the given VF f (or DS $\dot{x} = f$) can be of considerable help in the problem of normalizing the VF, as we shall briefly discuss now. A more complete discussion is given in [21], or in the paper [15] presented in the first edition of the SPT workshop.

As we did for the matrix A, we will assume that B satisfies Condition ω, and also that B is semisimple. A simple but useful result, based on the possible introduction of a convergent NT for the symmetry field g, is then:

Theorem 1. *Let the DS $\dot{x} = f = Ax + F$ admit an analytic symmetry $g = Bx + G(x)$ with B not proportional to A. Assume that either:* i) *the eigenvalues of B belong to the Poincaré domain, or* ii) *there is a coordinate transformation taking g into a NF $\widehat{g}$ satisfying Condition A (e.g. with a linear $\widehat{g} = Bx$). Then, there is a convergent coordinate transformation which takes (not only g into a NF but also) the DS into the form $f \to Ax + \widetilde{F}(x)$ where $\widetilde{F} \in \mathrm{Ker}(\mathcal{B})$, i.e. where $\widetilde{F}$ is in NF with respect to B (not necessarily to A).*

Remark 1. According to Theorem 3.1, another (possibly formal) coordinate transformation can put $\widetilde{F}$ into normal form also with respect to A (JNF), but it is not granted that this other transformation is convergent.

The above Theorem can be useful, e.g., in the special case where $\mathrm{Ker}(\mathcal{B}) = \{0\}$ (e.g. if $B = I$, see sect.3), and therefore in the problem of the linearization of the DS. In fact, results obtained in sect.3 can be now restated and completed in the following form[4, 26]:

Corollary 1. *A DS $\dot{x} = f = Ax + F$ can be linearized if and only if it admits a (possibly formal) symmetry $g = Bx + G$ such that $B = Dg(0) = I$; and the NT which linearizes f is convergent if and only if there is an analytic symmetry of this type. More in general, if g is analytic and $\mathrm{Ker}(\mathcal{B}) = \{0\}$, then the NT which linearizes f is convergent .*

Notice that, if g is analytic, the convergence of the NT is guaranteed by the Poincaré Theorem in the case $B = I$, and by Bruno Theorem (in particular, by Condition A on g) in the other general case $\mathrm{Ker}(\mathcal{B}) = \{0\}$.

Corollary 2. *With the same notations as in Corollary 1, if $\mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B}) = \{0\}$, and g is analytic, there is a convergent NT which takes f into*

$\mathrm{Ker}(\mathcal{A})$*, and another convergent NT which takes* f *into* $\mathrm{Ker}(\mathcal{B})$*. If* g *is linear then the linearizing transformation* $f \to Ax$*,* $g \to Bx$*, is convergent.*

8 Convergence without condition A

When Condition A is *not* satisfied by the VF X_f, the convergence of the NT of f can be equally ensured, under suitable conditions. These conditions are strictly related to the assumption that the given VF X_f, or the DS $\dot{x} = f(x) = Ax + F(x)$ admits some nontrivial *analytic* symmetry g.

One of the first results in this direction has been given by Markhashov[30] (his proof however appeared to be not complete[8]); recently, Bruno and Walcher[11] obtained a remarkable result in the case of 2-dimensional problems. Some recent results[12, 13, 14] on the problem of the convergence of NT, are related to and generalize the above mentioned ideas[11, 30]; here we just give a statement of the general result, referring to [14, 15] for details. Here the M_j represent a basis of the set of matrices commuting with A; one of these will always be taken to be indeed equal to A.

Theorem 1. *Given the analytic VF* f*, let us write its NF, according to Lemma* 4.3*, in the form*

$$\widehat{f} = Ax + \alpha(x)Ax + {\sum}' \mu_j(x)M_jx \equiv Ax + \alpha(x)Ax + \widehat{F}_1(x) \qquad (1)$$

with $\alpha(x)$*,* $\mu_j(x) \in \mathcal{I}_A^*$ *and (hereafter)* $\sum'$ *is the sum extended to the matrices* $M_j \neq A$*. Assume* $\widehat{F}_1(x) \neq 0$ *(otherwise Condition A is sufficient to ensure convergence of the NT), and:* i) *assume that* f *admits an analytic symmetry* $g = Bx + G(x)$ *such that* $B = aA$ *where* a *is a (possibly vanishing) constant, but* $G \neq aF$*;* ii) *assume that the equation* $\{\widehat{F}_1, S\} = 0$ *for the unknown power series* $S = S(x) = \sum' \nu_j(x)M_jx$ *with* $\nu_j(x) \in \mathcal{I}_A^*$ *and* $(DS)(0) = 0$ *has only the trivial solution* $S = c\widehat{F}_1(x)$ *[*$c =$ constant*]. Then* f *can be put into NF by means of a convergent NT.*

Remark 1. Assumption *ii)* of Theorem 1 implies that the NF $\widehat{f}$ admits only linear symmetries Lx. For a practical point of view (see examples in [12, 13, 14, 15]), it is quite simpler to verify the property *ii)*.

In the particular case that $\widehat{F}_1(x)$ in Theorem 1 has the special form $\widehat{F}_1(x) = \mu(x)Mx$ (with M not proportional to A), then assumption *ii)* is actually equivalent to the following one: there are no common (analytic, formal or fractional) constants of motion of the two linear problems $\dot{x} = Ax$ and $\dot{x} = Mx$.

This case has been considered in [12, 13], where the following theorems are obtained (theorem 4 is analogous to the Markhashov theorem[30]):

Theorem 2. *Assume that:* i) *the DS* (1) *admits an analytic symmetry* g *with the same assumptions as in* i) *of Theorem* 1*;* ii) *once in NF, the DS takes the form*

$$\dot{x} = Ax + \widehat{F}(x) = Ax + \alpha(x)Ax + \mu(x)Mx \qquad \alpha(x),\ \mu(x) \in \mathcal{I}_A^* \qquad (2)$$

where M is a matrix commuting with A but not proportional to A; iii) *the two linear problems $\dot{x} = Ax$ and $\dot{x} = Mx$ do not admit analytic nor formal nor fractional common constants of motion. Then, the DS can be put in NF by means of a convergent normalizing transformation.*

Corollary 1 [11]. *If a $2-$dimensional DS, with linear part A having eigenvalues λ_1, λ_2 satisfying $k_1\lambda_1 + k_2\lambda_2 = 0$ (with k_1, k_2 nonnegative relatively prime integers not simultaneously zero), admits an analytic symmetry g (with no other assumptions), then it can be normalized by a convergent transformation.*

Theorem 3. *Instead of* i) *in Theorem* 1 *assume that the DS admits $\ell \geq 1$ analytic symmetries $g_j = B_j x + G_j(x)$, such that no linear combination of them is proportional to f, where the matrices B_j are linearly independent, and where ℓ is precisely the number of the linearly independent* linear *symmetries admitted by the DS once in NF. Then, with the conditions* ii),iii) *as in Theorem* 2*, the same conclusion holds.*

Theorem 4. *Assume that the DS possesses a finite number $\ell \geq 1$ of analytic symmetries $g_j = B_j x + G_j(x)$, where all the matrices B_j are linearly independent. Assume also that, once in NF, the DS admits exactly ℓ linearly independent (possibly formal) symmetries. Then, there exists a convergent normalizing transformation for f.*

Theorem 4 appears to be less convenient for practical uses, in fact it requires the knowledge of *all* the symmetries of the NF. From this point of view, Theorems 1-3 may be more useful and concretely applicable. In fact, the greatest apparent difficulty is that, in general, one does not known – a priori – the NF: therefore, it seems to be impossible to check if the assumptions of these Theorems (or even Condition A) are verified by the NF. However, other symmetry properties of the VF may provide, as shown in the examples provided in [14, 15, 21], the decisive help on this point. To confirm the effectiveness of this remark, let us give a different statement of Theorem 2, where assumption *iii)* is conveniently replaced by another symmetry assumption.

Theorem 5. *Assume that f admits a symmetry g as in* i) *of Theorem* 1)*, and also a symmetry generated by a linear VF $Lx \cdot \nabla$ such that $\{g, Lx\} = 0$ and $\mathcal{I}^*_A \cap \mathcal{I}^*_M \cap \mathcal{I}^*_L = \mathbf{R}$ are satisfied. Then, if $\widehat{F}$ has the form* (2)*, the NT is convergent.*

Indeed, once f is put into NF, g preserves its property of being symmetric under Lx [28]; therefore, to verify assumption *iii)* of Theorem 2, it is enough to look for the simultaneous constants of motion of Ax, Mx, and Lx.

Remark 2. If all the eigenvalues of A are nondegenerate, the number ℓ of the linearly independent linear symmetries for the NF satisfies $1 \leq \ell \leq (n-1)$.

Remark 3. In the case of nondegenerate eigenvalues of the matrix A, the linear symmetries are clearly commuting; notice however that one may construct nonlinear symmetries which do not commute even if their linear parts are commuting. Instead, in the case of degenerate eigenvalues, the symmetries of the DS do not commute, in general (see examples, e.g., in [15, 19]).

For a full discussion of the relationship between symmetry properties of a DS (and of its NF) and convergence of the normalizing transformation, and for further results in this direction, we refer again to [15, 21].

Example 1. We will conclude this section – and this short review – by considering the following example, dealing with perturbation an m-dimensional isotropic oscillator.

Let $n = 2m$ and, putting $x \equiv (u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbf{R}^{2m}$; assume that a Lie group Γ acts "diagonally" on both the m-dimensional spaces of the vectors u and v through the same linear representation $\mathcal{D}$: i.e. $u \to u' = \mathcal{D}u$, $v \to v' = \mathcal{D}v$, where $\mathcal{D}$ is an absolutely irreducible representation (so the only matrix commuting with $\mathcal{D}$ is a multiple of the identity). Consider then a DS of the form $\dot{x} = f(x) = Ax + F(x)$, where

$$A \ = \ \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{3}$$

and assume that $F(x)$ admits the linear symmetries $L_i x$, where L_i are the (matrix representatives in the direct sum $\mathcal{D} \oplus \mathcal{D}$ of the) Lie generators of this group Γ $\big($the linear part Ax fulfils this symmetry requirement, so that the full DS admits this symmetry$\big)$. We shall choose, for instance, $m = 3$, $\Gamma = SO(3)$ and $\mathcal{D}$ its fundamental representation, and assume that the DS is

$$\dot{x} = Ax + p(x)x \tag{4}$$

where the scalar function p (cf. the example in [11]) is a homogeneous polynomial of degree $2k$ built with the quantities $u^2 = (u, u)$, $v^2 = (v, v)$ (where $(\cdot, \cdot)$ stands for the scalar product in $\mathbf{R}^3$). It is then easy to verify that the NF of the above DS must take the form

$$\dot{x} = Ax + \alpha Ax + \mu x \tag{5}$$

where α and μ are functions of $r^2 = u^2 + v^2$ only; on the other hand, one can see that the vector function (with vanishing linear part) $g = (u^2 + v^2)^k x$ generates a nontrivial nonlinear analytic symmetry for the DS, and that there are no common constants of motion, as requested by Theorem 2. Therefore, the convergence of the normalizing transformation is ensured. It is important to notice that, if our problem would not possess the symmetry Γ, the NF would contain many other terms in its r.h.s., and that it is precisely the presence of the symmetry which forces the NF to contain only A and the identity, and allows us to apply the argument concerning the constants of motion. This seems to confirm the conjecture[11] that the presence of a "sufficient" number of symmetries may be an essential request in order to guarantee the convergence of a normalizing transformation.

9 Further extensions

As mentioned in the introduction, we have tried to give here an overview of existing results on normal forms in the presence of nonlinear symmetries; lack of space prevents from considering further extensions of these results, but we would like to briefly mention in which direction these go.

First of all, as well known, when we deal with a Hamiltonian VF X_h generated by an Hamiltonian H, the NF of X_h can be chosen to preserve the Hamiltonian structure (i.e. it will be Hamiltonian itself), so that we can more economically normalize directly the (analytic) function H rather than the associated VF; the NFs for Hamiltonians are known as Birkhoff-Gustavsson NFs. The results we discussed here, in particular on JNFs and on relations between symmetries and convergence of the normalizing transformation, can be extended[21] – *mutatis mutandis* – to the Hamiltonian frame and Birkhoff-Gustavsson NFs. Now the homological operator will be defined in terms of the standard Poisson bracket, and the hypotheses related to the non-semisimple case would require some detailed discussion[21].

The formal results obtained in sect.3 can also be extended to the recently introduced *Poincaré renormalized forms*[24] in the presence of symmetries[25]. It is not known, so far, if the symmetry plays any role in the convergence of these.

Finally, some more recent results concerning the relations between convergence of the (Poincaré-Dulac) normalizing transformation, symmetry and existence of bifurcations are discussed in[16].

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