# FAMILIES OF RELATIVE EQUILIBRIA IN HAMILTONIAN SYSTEMS WITH DISSIPATION

GIANNE DERKS

*Dept. of Mathematics & Statistics, University of Surrey, Guildford, Surrey GU2 5XH, UK*

*E-mail: G.Derks@surrey.ac.uk*

In this note the influence of dissipation on families of relative equilibria in Hamiltonian systems will be considered. Relative equilibria can be described as critical points of an appropriate functional. This characterisation can be used to give sufficient conditions such that in finite dimensional systems with dissipation the extremal families of relative equilibria are stable under dissipation. Furthermore, a full class of families of relative equilibria in the Navier-Stokes equations will be analysed. For these families it will be shown that the extremal family of relative equilibria is an attractor and the non-extremal families of relative equilibria are unstable.

## 1 Introduction

A natural type of solutions in systems with (continuous) symmetries are solutions moving with the flow of the symmetry group. One type of such solutions are the relative equilibria. Relative equilibria are equilibria modulo the action of the symmetry group. For example, if the symmetry group is a rotation group, then the relative equilibria are uniformly rotating states.

Noether's Theorem implies that in Hamiltonian systems continuous symmetries are related to constant(s) of motion (they form the momentum map in mechanical systems). [1,28] Relative equilibria in Hamiltonian systems can be described as constrained critical points of the Hamiltonian on level sets of the constants of motion related to the symmetries. A Hamiltonian system can also have constants of motion which are not related to the symmetries. If the Poisson bracket of the Hamiltonian structure map has degeneracies, there will be constants of motion related to this degeneracy. These constants of motion are called Casimirs. [29] Equilibrium solutions of the Hamiltonian system which are constrained critical points of the Hamiltonian on level sets of the Casimirs, are called relative equilibria as well. Generically, relative equilibria in Hamiltonian systems will come in families which can be parametrised by the value of the level sets of the constants of motion. [32]

The variational description of the relative equilibria makes them accessible for detailed stability analysis in Hamiltonian systems. Two (related) systematic ways to analyse the stability of relative equilibria are the energy-Casimir method [2,3,20,23] and the (reduced) energy-momentum method. [35,36] The key to both these methods is that if the relative equilibria are constrained local extrema, then one can construct an appropriate Lyapunov-type functional based on the Hamiltonian and the extra constants of motion, and derive (orbital) stability. However, if the relative equilibria are not extremal, it is hard to draw any conclusion about stability or instability.

In many applications, the dynamics is described by a Hamiltonian system plus a (small) perturbation like dissipation and/or forcing. In this note we focus on the effects of weak equivariant dissipation on families of relative equilibria. Usually the constants of motion of the Hamiltonian system are not conserved anymore if dissipation is added to the system. So most relative equilibria will not be invariant, even modulo symmetries. Any trajectory must pass through the appropriate level sets of the constants of motion and will eventually leave the neighbourhood of the relative equilibrium it initially approximated and deviate far from this initial neighbourhood. Thus it is not generally useful to talk about the stability or instability of a single relative equilibrium, instead it is more appropriate to consider the full family of relative equilibria.

The basic questions which are considered in this note are the following

- If the family consists of relative equilibria which are constrained extrema, and a solution of the dissipative system starts near a relative equilibrium of the unperturbed system, can one sharply approximate it by a *time dependent curve of relative equilibria*?
- If the family is invariant and consists of relative equilibria which are not extremal, does the dissipation make the family truly unstable?

In section 2 a finite dimensional Hamiltonian system with dissipation is considered. The Hamiltonian system is symmetric and has a smooth, regular family of relative equilibria which are constrained extrema in the variational formulation. This implies that the relative equilibria are stable in the Hamiltonian system. [35,36] It will be shown that, under some reasonable hypotheses, solutions of the dissipative system which start near the family of relative equilibria can be shadowed by a curve of relative equilibria with dissipating momentum. A characterisation of this curve is given too. A simple example to which this method can be applied is a (rotating) spherical pendulum with friction.

If the family of relative equilibria is invariant under the dissipation too, usually the family becomes an attractor and the rate of attraction can be estimated with the method as described in section 2. Furthermore, the ideas sketched in section 2 can be extended to infinite dimensional Hamiltonian systems as well. These two extensions are illustrated in sections 3 and 4. In these sections the two dimensional Navier-Stokes equations are considered and it is shown that a family of stable stationary solutions of the incompressible homogeneous Euler equations becomes an attractor if viscosity is taken into account. Furthermore, a sharp estimate for the attraction rate will be derived.

In the last part of this note, we look at families of relative equilibria which are not constrained extrema in the variational formulation. These families can be either stable or unstable in the Hamiltonian system and in general it is hard to determine the (in)stability for the Hamiltonian system. In section 5, a Lyapunov-type criterion for the instability of invariant manifolds is derived. And in section 6 this criterion is invoked to show that families of non-extremal relative equilibria of the two dimensional Euler equations become unstable if viscosity is added, i.e., if one considers the two dimensional Navier-Stokes equations.

## 2 Stable families of relative equilibria in finite dimensional systems

In this section we consider a finite dimensional symplectic manifold $(\mathcal{M}, \omega)$, a compact group $G$ of Hamiltonian symmetries defining the smooth momentum map $\mathbf{J}$ and a $G$-invariant smooth Hamiltonian $H$. The Lie algebra, respectively the dual Lie algebra of the group $G$, are denoted by $g$ and $g^*$. And $\langle \cdot, \cdot \rangle$ is the pairing between the Lie algebra and its dual. The level sets of the momentum map are denoted by $\mathcal{M}_\mu$, i.e., for $\mu \in g^*$

$$\mathcal{M}_\mu = \{u \in \mathcal{M} \mid \mathbf{J}(u) = \mu\}. \tag{1}$$

We consider the following dynamical system on $\mathcal{M}$

$$\dot{u} = X_H(u) + \varepsilon P(u), \qquad u \in \mathcal{M}. \tag{2}$$

In this expression, $X_H : \mathcal{M} \to T\mathcal{M}$ is the Hamiltonian vector field, $P : \mathcal{M} \to T\mathcal{M}$ is a dissipative, $G$-equivariant smooth perturbation and $\varepsilon$ is a small parameter which measures the strength of the perturbation. The state $\overline{u} \in \mathcal{M}$ is called the shape of a relative equilibrium if there exists a $\xi \in g$ such that $\exp(t\xi)\overline{u}$ is a solution of the Hamiltonian system ($\varepsilon = 0$). Here $\exp(\xi)$ is the group flow generated by $\xi \in g$.

It is assumed that the unperturbed Hamiltonian system possesses a smooth connected manifold of shapes of relative equilibria, which can be parametrised by the value of the momentum map. To be specific, there is a subset $h^* \subset g^*$ such that for all $\mu \in h^*$ there is at least one relative equilibrium with momentum $\mu$. Furthermore, if $\overline{u}$ is the shape of a relative equilibrium, then the orbit of this relative equilibrium is equal to the set of all relative equilibria in the manifold with momentum $\mu$. So we can introduce maps $\overline{u}$ from $h^*$ to the manifold of relative equilibria and $\overline{\xi} : h^* \to g$ such that for each $\mu \in h^*$, $\overline{u}(\mu)$ is a relative equilibrium with momentum $\mu$ and generator $\overline{\xi}(\mu)$.

It is also assumed that the group $G$ acts freely in a neighbourhood of the family of relative equilibria and that $\mathbf{J}$ is regular at the family of relative equilibria. A sufficient condition for this is that $D\mathbf{J}(\overline{u})$ is surjective for all $\overline{u}$. This condition implies also that $T_{\overline{u}(\mu)}\mathcal{M}_\mu = \ker[D\mathbf{J}(\overline{u}(\mu))]$.

Since $\overline{u}(\mu)$ is the shape of a relative equilibrium, $\overline{u}(\mu)$ is a critical point of the energy-momentum function or augmented Hamiltonian [35]

$$H_\mu : \mathcal{M} \to \mathbf{R}, \quad H_\mu = H - J_{\xi(\mu)}, \quad \text{with} \quad J_\xi(u) = \langle J(u), \xi \rangle. \tag{3}$$

Furthermore it is assumed that for all relative equilibria, the second derivative of $H_\mu$ is semi-positive definite on $T_{\overline{u}(\mu)}\mathcal{M}_\mu = \ker[D\mathbf{J}(\overline{u}(\mu))]$, with kernel the tangent vector fields generated by the Lie algebra, i.e., $\{X_{J_\xi}(\overline{u}) \mid \xi \in g_\mu\}$. This condition ensures that the relative equilibria are constrained minima, hence orbitally stable solutions of the unperturbed Hamiltonian system ($\varepsilon = 0$).

To analyse the perturbed system, define the following Lyapunov functional

$$L(u) = H(u) - H(\overline{u}(\mu)) = H_\mu(u) - H_\mu(\overline{u}(\mu)), \quad \text{where} \quad \mu = \mathbf{J}(u). \tag{4}$$

For $u$ near the family of relative equilibria, this functional measures the distance between $u$ and the orbit of the relative equilibrium with shape $\overline{u}(\mathbf{J}(u))$. [9] This distance is denoted by $d(u, \overline{u})$.

If one calculates the time derivative of the functional $L$, then one sees that there are two important opposing effects of the dissipation on the family of relative equilibria. First there is an attracting effect, which pulls solutions towards the family. Secondly, if the family is not invariant, then there is a repelling effect, which pushes solutions away from the family of relative equilibria. If the family is invariant, then this effect reduces to the effect of the curvature of the manifold, which can be expressed in terms of the perturbation at the manifold of relative equilibria. [9,11]

An important observation is that if the perturbation $P(u)$ has a component in the direction of vector fields generated by the Lie algebra $g_\mu$, then this component has only effects on the $G_\mu$-orbit of the solution, where $\mu = \mathbf{J}(u)$. Hence the only relevant part of the perturbation is the part perpendicular to the vector fields generated by the Lie algebra $g_\mu$. This part is denoted by $\widehat{P}(u)$.

There are three technical hypotheses on the perturbation, which specify its dissipative character and quantify the effects of the dissipation mentioned above. First, the influence of the perturbation on the momentum map has to be such that the value of the momentum map $\mathbf{J}(u(t))$ of any solution $u(t)$ has a limit for $t \to \infty$. So, although it is not necessary that all solutions are bounded, it is assumed that the momentum map stays within a bounded set of values for any solution. This implies that we have to look at a compact subset of $g^*$ only. This is a quite natural condition for a dissipative system, for which one expects that all physical quantities settle down to a limiting value. The time evolution of the momentum map along a solution is given by

$$\frac{d}{dt}\mathbf{J}(u) = \varepsilon D\mathbf{J}(u) \cdot P(u). \tag{5}$$

Hence this is a function of a slow time variable $\tau = \varepsilon t$.

Secondly, every relative equilibrium has to be attractive in a "quasi-static" context. This means that the linearisation of the perturbation in directions tangent to the level set of the momentum map is attractive towards the relative equilibria. In first approximation, the tangential dissipation coefficient is the determining factor for this behaviour. The tangential dissipation coefficient $\beta_T(\mu)$ is defined as

$$\beta_T(\mu) = -\max_{\delta u \in S(\overline{u}(\mu))} \frac{D^2 H_\mu(\overline{u}(\mu))\,(\,\delta u\,,\, D\widehat{P}(\overline{u}(\mu))\,\delta u\,)}{D^2 H_\mu(\overline{u}(\mu))\,(\delta u\,,\,\delta u)}, \tag{6}$$

where $S(\overline{u})$ is the tangent space to $\mathcal{M}_\mu$ excluding the directions of vector fields generated by the Lie algebra $g_\mu$. This definition shows that $\beta_T$ measures the effect of the dissipation along a level set $\mathcal{M}_\mu$. To obtain an accurate value for the dissipation coefficient, it is sometimes necessary to split the perturbation in a truly dissipative part and a Hamiltonian part. [8] In the example below this will be illustrated.

Finally, the manifold of relative equilibria need not be invariant under the perturbed dynamics, but the effects of the perturbation that push trajectories away from the manifold of relative equilibria should not be strong for a long time. The effect of the perturbation at the manifold is $\varepsilon\widehat{P}(\overline{u}) - \dot{\overline{u}}$. For many dissipative systems, the evolution of the momentum map as given in (5) induces an exponential decay to the limiting value of the momentum map. Then we can expect that $P(\overline{u})$, $\overline{u}$ and $\dot{\overline{u}}$ are exponentially decaying to their limiting values, since all these quantities are parametrised by the momentum map. So it is assumed that there exists some $\alpha > 0$, a constant $C$ and some integrable function $k(\tau)$ such that

$$|\widehat{P}(\overline{u})| \leq Ce^{-\varepsilon\alpha t}, \quad \text{and} \quad |\varepsilon\widehat{P}(\overline{u}) - \dot{\overline{u}}| \leq \varepsilon k(\varepsilon t)e^{-\alpha\varepsilon t}. \tag{7}$$

Another effect that can push solutions away from the manifold, even if the manifold is invariant, is the curvature of the manifold. The evolution along the level sets is described by the $\mu$ equation, hence usually the tangential dissipation coefficient $\beta_{\mathrm{T}}(\mu)$ is the most relevant term to describe the repelling effects. If the limiting value of the momentum map is a regular value of $\mathbf{J}$ and $H$ is sufficiently smooth, then the effects of the curvature are captured by the condition (7) and the tangential dissipation coefficient. If this is not the case, then we have to add some extra condition, see [9] for details.

If the hypotheses on the dissipation as described above are satisfied, the curve $\overline{u}(\mu(t))$ shadows that solution $u(t)$ for solutions $u(t)$ starting near the manifold $\mathcal{M}$. This is made explicit in the following theorem [9]

**Theorem 1** *Let $u(t)$ be a solution of the dissipative Hamiltonian system (2). Define $\mu(t) = \mathbf{J}(u(t))$ and $\beta = \lim_{t\to\infty} \beta_T(\mu(t))$. If $u(0)$ is close to the shape of a relative equilibrium in the level set $\mathcal{M}_{\mu(0)}$, then for all $t \geq 0$, $u(t)$ stays close to the shape of a relative equilibrium in the level set $\mathcal{M}_{\mu(t)}$. To be precise, if $\varepsilon$ is sufficiently small then*

$$L(u(0)) = \mathcal{O}(\varepsilon^2) \quad \textit{implies} \quad L(u(t)) = \mathcal{O}(\varepsilon^2\, e^{-2\min(\alpha,\beta)\varepsilon t}), \qquad t \geq 0. \tag{8}$$

*Or by using the distance $d$, if*

$$d(u(0), \overline{u}(\mu(0))) = \mathcal{O}(\varepsilon), \qquad \textit{then} \qquad d(u(t), \overline{u}(\mu(t))) = \mathcal{O}(\varepsilon\, e^{\min(\alpha,\beta)t}), \tag{9}$$

*for all $t \geq 0$.*

One of the questions that comes up if one sees this result (and its proof) is whether in the dissipative system there would be an invariant manifold, which is a perturbation of the family of relative equilibria. If it exists, this invariant manifold would be an attractor. The family of relative equilibria is stable in the Hamiltonian system and hence does not have any hyperbolic properties, so results as in [15,19] can not be used immediately. In [18,37] finite dimensional integrable Hamiltonian systems with dissipative perturbation are considered. These papers give conditions on the dissipation such that a weakly attractive KAM torus persists and how to numerically calculate such a torus. It would be interesting to see if these results can be combined with the work mentioned above to give persistence of the family of relative equilibria.

To illustrate Theorem 1 on shadowing with curves on the manifold of relative equilibria, we will look at a simple example in $\mathbf{R}^4$. The Hamiltonian system in this example is integrable. However, this property is irrelevant for the method, which does not assume any integrability properties; it is a consequence of the low dimensionality.

**Example 1** We consider a spherical pendulum with a pivot, which is rotating with constant angular momentum $\Omega$. The gravity is denoted by g, the length of the pendulum by $l$ and the pendulum is slightly damped due to friction. We introduce spherical coordinates $\varphi$ and $\theta$, where $\theta$ is the angle with the vertical, see Figure 1, and $\varphi$ is the angle in the horizontal plane (perpendicular to the gravity) in a frame rotating with the pivot.

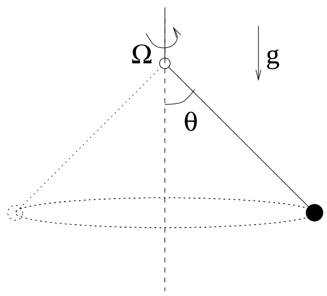


Figure 1. A spherical pendulum with coordinates $\theta$ and $\varphi$.

After introducing the impulses $p_\varphi$ and $p_\theta$, we get the following Hamiltonian system with dissipation

$$\begin{aligned} \dot{\varphi} &= \tfrac{p_\varphi}{\sin^2\theta} - \Omega \\ \dot{\theta} &= p_\theta \\ \dot{p}_\varphi &= 0 - \varepsilon\left(p_\varphi - \Omega \sin^2\theta\right) \\ \dot{p}_\theta &= \tfrac{p_\varphi^2 \cos\theta}{\sin^3\theta} - \tfrac{g}{l}\sin\theta - \varepsilon p_\theta. \end{aligned} \tag{10}$$

Hence the dynamics takes place on the manifold $\mathcal{M} = S^2 \times \mathbf{R}^2$, with $u = (\varphi, \theta, p_\varphi, p_\theta) \in \mathcal{M}$. The symplectic form $\omega$ is the canonical symplectic form. The Hamiltonian is the sum of potential and kinetic energy

$$H(u) = \frac{1}{2}p_\theta^2 + \frac{1}{2}\frac{p_\varphi^2}{\sin^2\theta} - \Omega p_\varphi - \frac{g}{l}\cos\theta. \tag{11}$$

The system is equivariant under rotations in the plane, hence the symmetry group is $G = S^1$ and we can identify the Lie algebra $g$ and its dual $g^*$ with $\mathbf{R}$. The momentum map is the angular momentum $\mathbf{J}(u) = p_\varphi$.

The family of shapes of relative equilibria is given by

$$\left\{\overline{u}(\mu) = \left(\varphi, \overline{\theta}(\mu), \mu, 0\right) \mid \varphi \in S^1,\, \mu \in \mathbf{R}^+\right\} \tag{12}$$

with $\overline{\theta}(\mu)$ the solution of $\mu^2\cos\theta = \frac{g}{l}\sin^4\theta$. Thus regular relative equilibria exist for $\mu \in h^* = \mathbf{R}^+ \subset g^*$. The generator $\xi(\mu) = \frac{\mu}{\sin^2\overline{\theta}} - \Omega$. So in the Hamiltonian system ($\varepsilon = 0$), the relative equilibria are solutions in which the pendulum rotates at a fixed height with a constant angular velocity. The augmented Hamiltonian is

$$H_\mu(u) = \frac{1}{2}p_\theta^2 - \frac{\mathrm{g}}{l}\cos\theta + \frac{1}{2}\frac{p_\varphi^2}{\sin^2\theta} - \frac{p_\varphi \mu}{\sin^2(\overline{\theta}(\mu))}. \tag{13}$$

It is straightforward to verify that the shapes of the relative equilibria $(\overline{u}(\mu))$ are constraint minima of the Hamiltonian on level sets of the angular momentum $\mathbf{J}(u) = \mu$.

The Lyapunov functional for the family is

$$L(u) = H(u) - H(\overline{u}(u)), \tag{14}$$

where $\overline{u}(u)$ is the relative equilibrium with $\mathbf{J}(\overline{u}) = \mathbf{J}(u)$. So $\overline{u}(u)$ is a projection of the state $u$ on the family of relative equilibria. This projection will give the shadowing curve in the dissipative system.

Next we consider the influence of friction. The family of relative equilibria is not invariant under friction, only the two shapes with $\mu = 0$ ($\overline{\theta} = 0$) and $\mu = \mu_0 = \frac{\Omega^4 l^2 - \mathrm{g}^2}{\Omega^3 l^2}$ ($\overline{\theta} = \arccos\frac{\mathrm{g}}{l\Omega^2}$) represent invariant relative equilibria. We will look at the case that $\mathrm{g} < l\Omega^2$, hence there are two invariant relative equilibria. A short calculation shows that the zero state is unstable in this case.

First we analyse the behaviour of the momentum map. The first order approximation near the family of relative equilibria is given by

$$\dot{\mu} = -\varepsilon\,[\,\mu - \Omega\sin^2(\overline{\theta}(\mu))\,] = -\varepsilon\mu\,[\,1 - \Omega\sqrt{\tfrac{l}{\mathrm{g}}}\sqrt{\cos(\overline{\theta}(\mu))}\,]. \tag{15}$$

It can be shown that every solution of this equation is attracted to $\mu = \mu_0$. And $\mu_0$ is a regular value of the momentum map $\mathbf{J}$. Furthermore, the attraction rate near $\mu = \mu_0$ is like $e^{-\varepsilon\frac{\Omega^4 l^2 - \mathrm{g}^2}{3\mathrm{g}^2 + \Omega^4 l^2}t}$. The attraction rate for the momentum map $\mu(t)$ as given by (15) is a monotonic decaying function in $\frac{\mathrm{g}}{\Omega^2 l}$. If $\frac{\mathrm{g}}{\Omega^2 l}$ goes from 0 to 1, then the attraction rate changes from $e^{-\varepsilon t}$ to 1.

Next we calculate the tangential dissipation coefficient, which measures the dissipative effect of the perturbation. A short calculation shows that $\beta_T(\mu) = 0$. This is caused by the fact that the dissipation only acts on the $p_\theta$ variable and not on $\theta$ itself. However, it is known that friction has effect both on the impulse $p_\theta$ and the coordinate $\theta$. To see this effect, we add a Hamiltonian part of the perturbation to the Hamiltonian and only keep the dissipative effects (see also [8]). We write

$$P = -\frac{\varepsilon}{2}(0, \theta - \overline{\theta}(\mu_0), 2p_\varphi, p_\theta)^T + \frac{\varepsilon}{2}(0, \theta - \overline{\theta}(\mu_0), 0, -p_\theta)^T = P_{\mathrm{new}} + P_{\mathrm{Ham}}. \tag{16}$$

The first part of this expression is dissipative and the second part is Hamiltonian and will be added to the Hamiltonian. This split has an order 1 effect on the perturbation, but order $\varepsilon$ or smaller on the Hamiltonian and relative equilibria (the new relative equilibria $\overline{u}_{\rm new}(\mu)$ is order $\varepsilon^2|\mu-\mu_0|$ close to the original one). With this new expression for the perturbation, we get that $\beta_T(\mu)=\frac{1}{2}$, hence $\beta=\frac{1}{2}$.

Finally we look at the effect of the dissipation at the non-invariant family of relative equilibria. Define $\alpha_0=\frac{\Omega^4 l^2-{\rm g}^2}{3{\rm g}^2+\Omega^4 l^2}$, then we have for any $0<\alpha<\alpha_0$

$$|\widehat{P}_{\rm new}(\overline{u}_{\rm new})|\leq Ce^{-\varepsilon\alpha_0 t}\quad\text{and}\quad|\varepsilon\widehat{P}_{\rm new}(\overline{u}_{\rm new})-\dot{\overline{u}}_{\rm new}|\leq\varepsilon k(\varepsilon t)e^{-\alpha\varepsilon t},\qquad(17)$$

with $k(\varepsilon t)$ an integrable function, $C$ some constant and $\overline{u}_{\rm new}(t)$ is a smooth curve of relative equilibria with $\mathbf{J}(\overline{u})=\mathbf{J}(u)$.

We can conclude that the conditions on the perturbation as quoted in Theorem 1 are satisfied and we get

**Theorem 2** *Let $u(t)$ be a solution of (10) with ${\rm g}<l\Omega^2$ and let $\overline{\theta}_0$ be the $\theta$-angle of a relative equilibrium such that $\overline{p}_\varphi(\overline{\theta}_0)=p_\varphi(0)$. If $|\theta(0)-\overline{\theta}_0|^2+|p_\theta(0)|^2=\mathcal{O}(\varepsilon^2)$, then with $\overline{\theta}_t$ $\theta$-angles of a relative equilibria such that $\overline{p}_\varphi(\overline{\theta}_t)=p_\varphi(t)$,*

$$|\theta(t)-\overline{\theta}_t|^2+|p_\theta(t)|^2=\mathcal{O}(\varepsilon^2 e^{-2\varepsilon\gamma t}),\qquad(18)$$

*for all $t\geq 0$, with $\gamma=\min(\alpha,\beta)$ and any $0<\alpha<\alpha_0=\frac{\Omega^4 l^2-{\rm g}^2}{3{\rm g}^2+\Omega^4 l^2}$.*

Hence if the spherical pendulum starts its evolution nearby a relative equilibrium solution, the evolution can be shadowed by a curve of relative equilibria and the difference between the solution and its approximation becomes exponentially small. In the limit for $t\to\infty$, the spherical pendulum evolves like a relative equilibrium with angular momentum $\mu=\mu_0=\frac{\Omega^4 l^2-{\rm g}^2}{\Omega^3 l^2}$. The angular momentum decays exponentially fast to this value, as can be seen from (15).

In Figure 2, we have sketched some solutions of the spherical pendulum, projected in the horizontal $x$-$y$ plane. The initial state is a relative equilibrium. In the left picture we start with $\theta(\mu)=0.1$ and in the right picture with $\theta(\mu)=0.9$. ◇

More examples of applications of the general method sketched above (also with non-Abelian symmetry groups) can be found in [8,9]. An extension to an infinite dimensional Hamiltonian system with a one dimensional symmetry group is given in [10]. In this paper, the approximation of solutions of the uniformly damped periodic Korteweg-de Vries equation with a curve of cnoidal waves (which are solitary-like solutions of the periodic KdV equation) is considered.

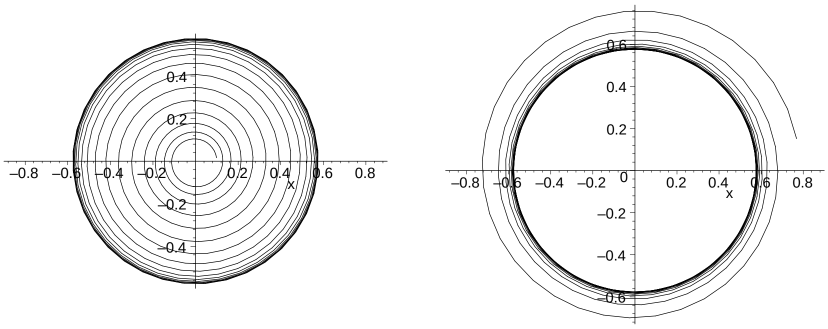


Figure 2. Projection on the horizontal plane of the orbit of a damped spherical pendulum, starting at a relative equilibrium with $\overline{\theta}(0) = 0.1$ (left) respectively with $\overline{\theta}(0) = 0.9$ (right).

## 3 Relative equilibria in 2D Navier-Stokes equations

In this section we will look at a Hamiltonian pde with dissipation. Consider an incompressible flow in a two dimensional compact domain $\mathcal{D}$ in $x$-$y$ plane, or on a sphere with radius $R$, see Figure 3.

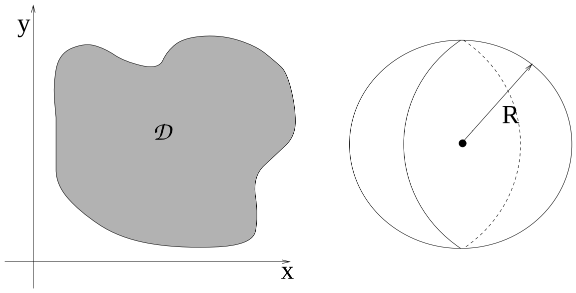


Figure 3. The domain for the incompressible flow.

Because of the incompressibility condition the velocity $\mathbf{v}(\mathbf{x})$ is divergence free, so a stream function $\psi(\mathbf{x})$, with $\mathbf{x} \in \mathcal{D}$ or on the sphere, can be defined. Furthermore, the vorticity is denoted by $\omega$. By definition, $\omega = \operatorname{curl} \mathbf{v}$ and $\omega = -\Delta\psi$. In the vorticity formulation, the two dimensional Navier-Stokes equations (NS) take the following form

$$\omega_t = -\mathbf{v} \cdot \nabla\omega + \nu\Delta\omega. \tag{19}$$

Here $\nu$ represents the viscosity. For the sphere no boundary conditions are needed. For the domain $\mathcal{D}$, the following boundary condition

$$\omega = 0 \quad \text{on} \quad \partial\mathcal{D} \tag{20}$$

is imposed. If $\mathcal{D}$ is periodic, one can also use periodic boundary conditions.

In case there is no vorticity ($\nu = 0$), the flow behaves like an ideal fluid and can be described by Euler equations. These form a system of Hamiltonian pde's. The Hamiltonian is the kinetic energy

$$H(\omega) = \frac{1}{2}\int_{\mathcal{D}} |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} = \frac{1}{2}\int_{\mathcal{D}} \omega(\mathbf{x})\psi(\mathbf{x}) \, d\mathbf{x}. \tag{21}$$

Furthermore, the Poisson map is degenerate (the Hamiltonian form of Euler equations in the plane is $\dot{\omega} = (-\nabla\omega \cdot J\nabla)DH(\omega)$, with $J$ the standard canonical matrix in $\mathbf{R}^2$.) This degeneracy gives rise to many Casimir functionals, for example all functionals depending on the vorticity $\omega$ only

$$C_f(\omega) = \int_{\mathcal{D}} f(\omega(\mathbf{x})) \, d\mathbf{x}, \tag{22}$$

with $f$ an arbitrary smooth function. In this note, only the enstrophy

$$W(\omega) = \frac{1}{2}\int \omega^2(\mathbf{x}) \, d\mathbf{x} \tag{23}$$

will be used. Note that both the Hamiltonian and the enstrophy are norms for the vorticity.

Using the enstrophy, special stationary state solutions which are relative equilibria can be found. I.e., these solutions are constrained critical points of the Hamiltonian on level sets of the enstrophy. The critical point problem

$$\operatorname{crit}\{H(\omega) \mid W(\omega) = w\} \tag{24}$$

leads to many families of relative equilibria, which are denoted by $\mathcal{E}_k$. The state $\overline{\omega} \in \mathcal{E}_k$ if and only if

$$-\Delta\overline{\omega} = \lambda_k\overline{\omega}, \tag{25}$$

with $\lambda_k$ an eigenvalue of $-\Delta$ on $\mathcal{D}$ with vanishing boundary conditions and

$$0 < \lambda_0 < \lambda_1 < \ldots. \tag{26}$$

Hence the Euler-Lagrange equation is the eigenvalue problem for the Laplace operator on $\mathcal{D}$ and the Lagrange multiplier is the inverse of the eigenvalue. Each eigenvalue gives a different families and the relative equilibria in each family are parametrised by the value of the enstrophy. These relative equilibria are like two-dimensional ABC-flows, they arise in a similar way as the original ABC-flows in 3D flow. [13] In case of a compact domain $\mathcal{D}$, the family

$\mathcal{E}_0$ is always one dimensional. [4] In case $\mathcal{D}$ is a sphere, the family $\mathcal{E}_0$ is three dimensional. [7]

Some examples of relative equilibria are sketched in Figure 4. In this figure the domain $\mathcal{D}$ is a square. In the left picture the vorticity of a relative equilibrium in the family $\mathcal{E}_0$ is sketched, in the right picture the vorticity of a member of the family $\mathcal{E}_1$. The family $\mathcal{E}_0$ is one-dimensional, so all other relative equilibria have the same shape as the sketched one but are scaled in amplitude. The greater the value of the enstrophy in the critical point problem (24), the higher is the amplitude of the relative equilibrium. The family $\mathcal{E}_1$ is two-dimensional. Scalings and a rotation about 90° will give all other relative equilibria in this family.

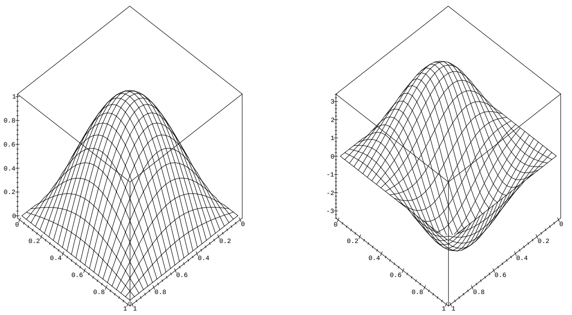

Figure 4. Sketch of the vorticity of a relative equilibrium in $\mathcal{E}_0$ respectively $\mathcal{E}_1$ when $\mathcal{D}$ is a square.

The Lyapunov functional for the family $\mathcal{E}_k$ is

$$L_k(\omega) = W(\omega) - \lambda_k H(\omega) = -\lambda_k(H(\omega) - H(\widehat{\omega})) = W(\omega) - W(\widetilde{\omega}), \qquad (27)$$

where $\widehat{\omega}$ and $\widetilde{\omega}$ are relative equilibria in $\mathcal{E}_k$ with $W(\widehat{\omega}) = W(\omega)$ respectively $H(\widetilde{\omega}) = H(\omega)$. The functional $L_0$ is a semi-norm. Indeed, for any relative equilibrium $\overline{\omega} \in \mathcal{E}_0$ and any vorticity $\xi$ we have

$$L_0(\overline{\omega} + \xi) = \frac{1}{2}\int \xi(\xi - \lambda_0\varphi) \equiv \|\Pi_0^{\perp}\xi\|^2, \quad \text{with} \quad -\Delta\varphi = \xi, \qquad (28)$$

where $\Pi_0$ denotes the $L^2$-orthogonal projection onto $\mathcal{E}_0$ and $\Pi_0^{\perp} = I - \Pi_0$. Hence any relative equilibrium in $\mathcal{E}_0$ is a minimum of $L_0$ (constrained maximum of $H$) and $L_0(\omega)$ measures the distance between $\omega$ and the family $\mathcal{E}_0$. On the other hand, the functionals $L_k$ are not positive definite for $k \geq 1$. If $\overline{\omega} \in \mathcal{E}_0$ then $L_k(\overline{\omega}) = (\lambda_0 - \lambda_k)H(\overline{\omega}) < 0$ and if $\overline{\omega} \in \mathcal{E}_{k+1}$, then $L_k(\overline{\omega}) = (\lambda_{k+1} - \lambda_k)H(\overline{\omega}) > 0$. Hence for $k \geq 1$, the families $\mathcal{E}_k$ are not extremal.

So according to the energy-Casimir method, it can be concluded that the relative equilibria in $\mathcal{E}_0$ are stable under the Euler equations. [2,3,20] However, no conclusion can be drawn about the other families and their stability under the Euler equations is unknown.

After finding the relative equilibria and analysing their Lyapunov functionals, we return to the Navier-Stokes equations. All families $\mathcal{E}_k$ are invariant in Navier–Stokes equations too. However, the individual relative equilibria are not invariant anymore. Instead there is an evolution along the family, the amplitude of the relative equilibria is exponentially decaying. For any $\overline{\omega} \in \mathcal{E}_k$, $k \in \mathbf{N}_0$, the evolution under the Navier-Stokes equations is

$$\omega(t) = \overline{\omega} e^{-\nu\lambda_k t}. \tag{29}$$

Furthermore, for any solution of the Navier-Stokes equations, the Hamiltonian and the enstrophy are at least exponentially decaying ($\dot{H} \leq -2\nu\lambda_0 H$ and $\dot{W} \leq -2\nu\lambda_0 W$). So neither constant of motion survives the influence of the viscosity term.

In the next section it will be shown that the family $\mathcal{E}_0$ is an attractor for the Navier-Stokes equations by using a similar approach as for the finite dimensional Hamiltonian systems in section 2. This gives a simple proof and a slight improvement of results known in literature. [14,16] Similar techniques as presented in this note for the Navier-Stokes equations can be used to analyse other equations like viscous reduced magnetohydrodynamic equations. [11] In section 6 it will be shown that the other families $\mathcal{E}_k$, $k \geq 1$, are unstable under the Navier-Stokes equations.

## 4 Stability of the family $\mathcal{E}_0$

In the previous section it is shown that the family $\mathcal{E}_0$ is stable under the Euler equations and that the functional $L_0$ is a semi-norm, measuring the distance to $\mathcal{E}_0$. In order to use this fact under the Navier-Stokes equations, a Hamiltonian projection of the vorticity onto the family $\mathcal{E}_0$ is defined. This is necessary because the level sets of the enstrophy ($W$) and the Hamiltonian ($H$) change under the Navier-Stokes equations in contrary to under the Euler equations.

Let $\omega$ be some vorticity. Define the projection $\overline{\omega}(\omega)$ of $\omega$ onto $\mathcal{E}_0$

$$\overline{\omega}(\omega) = \sqrt{2W(\omega)}\,\mathrm{sgn}\left(\int_{\mathcal{D}} \omega\Omega_0 d\mathbf{x}\right)\,\Omega_0, \tag{30}$$

where $\Omega_0$ is a normalised element of $\mathcal{E}_0$, $||\Omega_0|| = 1$. This implies that $\overline{\omega}(\omega) \in \mathcal{E}_0$ and $W(\overline{\omega}(\omega)) = W(\omega)$. To measure the distance between $\omega$ and its projection $\overline{\omega}$, we will use $L_0$. Since every solution decays to the zero state, we introduce a weighted Lyapunov functional in order to get more accurate estimates for the difference in shape. The weighted functional is

$$\widehat{L}_0(\omega) = \frac{L_0(\omega)}{H(\omega)} = \frac{W(\omega) - \lambda_0 H(\omega)}{H(\omega)}. \tag{31}$$

Since $H$ is a norm for the vorticity, $\widehat{L}_0$ is indeed a weighted functional (the division by $H$ neutralises the influence of the amplitude). Furthermore, since $H$ is a constant of motion for the Euler equations, $\widehat{L}_0$ is a Lyapunov functional for the Euler equations as well as $L_0$.

Next we define a cone around the family $\mathcal{E}_0$

$$\mathcal{B} = \{\, \omega_0 \mid W(\omega_0) < \lambda_1 H(\omega_0)\,\}, \tag{32}$$

see Figure 5. By using some Poincaré inequalities, the time derivative of

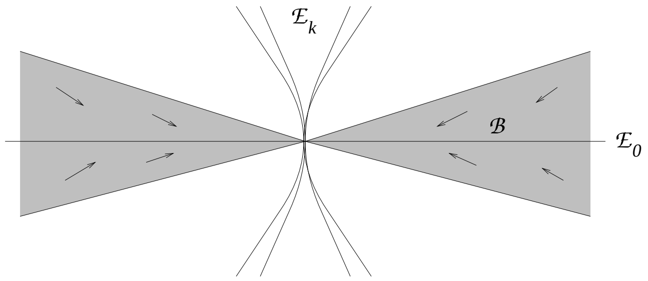


Figure 5. The cone $\mathcal{B}$ is within the basin of attraction of $\mathcal{E}_0$.

$\widehat{L}_0$ for any solution of the Navier-Stokes equations starting within $\mathcal{B}$ can be estimated (see [11]). This leads to

$$\frac{d}{dt}\widehat{L}_0(\omega(t)) \le -2\nu(\lambda_1 - \lambda_0)\widehat{L}_0 + 2\nu\widehat{L}_0^2, \tag{33}$$

which can be rewritten as

$$\frac{d}{dt}\left[\frac{1}{e^{2\nu(\lambda-\lambda_0)t}\widehat{L}_0(\omega(t))}\right] \ge -2\nu e^{-2\nu(\lambda-\lambda_0)t}. \tag{34}$$

This inequality implies that if a solution enters $\mathcal{B}$, it will stay in $\mathcal{B}$ for all time. Hence $\mathcal{B}$ is contained in the basin of attraction of $\mathcal{E}_0$. Furthermore, all states with vorticity within the cone $\mathcal{B}$ satisfy $\widehat{L}_0 < \lambda_1 - \lambda_0$. Note that that states on $\mathcal{E}_1$ satisfy $\widehat{L}_0 = \lambda_1 - \lambda_0$. Since $\mathcal{E}_1$ is invariant and "far away" from $\mathcal{E}_0$ (in the distance induced by $\widehat{L}_0$), $\mathcal{B}$ is a maximal possible cone in this sense.

The estimate on the time derivative of $\widehat{L}_0$ (34) can be used to derive an estimate for the rate with which solutions are attracted to $\mathcal{E}_0$. [11]

**Theorem 3** *The set of all Eulerian relative equilibria*

$$\mathcal{E}_0 = \{\, \overline{\omega} \mid \overline{\omega} \text{ is a } \lambda_0\text{-eigenvector of } (-\Delta)\,\} \tag{35}$$

*is an attractor for Navier-Stokes equations. The basin of attraction contains the cone $\mathcal{B}$, defined in (32).*

*If $\omega(t)$ is a solution of the Navier-Stokes equations with $\omega(0) \in \mathcal{B}$, then*

$$\frac{\|\Pi_0^{\perp}\omega(t)\|}{\|\Pi_0\omega(t)\|} \leq C(\omega(0))e^{-\nu(\lambda_1-\lambda_0)t} \tag{36}$$

*for all $t \geq 0$. Here $C$ is a constant which depends on the initial vorticity $\omega_0$ only. Furthermore, $\|\cdot\|$ is the $L^2$-norm on $\mathcal{D}$ and $\Pi_0$ denotes the $L^2$-orthogonal projection onto $\mathcal{E}_0$, hence $\Pi_0^{\perp} = I - \Pi_0$.*

## 5 Unstable families of relative equilibria

So far the variational description of the relative equilibria has been used to obtain stability results in Hamiltonian equations with dissipation. In [5,6] a relative equilibrium which is not a constrained extremum is considered. It is shown that by adding a small, momentum preserving dissipation to the Hamiltonian system the relative equilibrium becomes unstable. In the first paper [5] it is shown that momentum preserving friction in a simple mechanical system will lead to instability for non-extremal relative equilibria and in the second paper [6] the Euler-Poincaré equations with double bracket dissipation are considered and again nonlinear instability for non-extremal relative equilibria is derived. However, in both papers only dissipations which conserve the relevant constants of motion are considered.

There exist Lyapunov-type theorems both for stability and instability. [17,26,30,38] These theorems refer to the instability of fixed points. To investigate the instability of families of non-extremal relative equilibria under the influence of dissipations which do not conserve the constants of motion related to the symmetries, one has to extend these Lyapunov-type theorems for instability of fixed points to invariant manifolds. In this section one such extension will be presented.

Consider a smooth dynamical system

$$\frac{du}{dt} = f(u,t), \tag{37}$$

with $u$ in some metric spaces, with distance functions $d_1$ and $d_2$. Let $\mathcal{M}$ be an invariant manifold of the equation. The following lemma gives sufficient conditions for the instability of this manifold by using an indefinite Lyapunov functional.

**Lemma 4** *Let $L$ be a smooth functional which satisfies:*

1. *At the manifold $\mathcal{M}$ the functional $L$ vanishes (for any $u \in \mathcal{M}$ it holds $L(u) = 0$);*
2. *In any neighbourhood of $\mathcal{M}$ there are states at which $L$ is negative (for all $\delta > 0$ there is some $u$ with $d_1(u, \mathcal{M}) < \delta$ and $L(u) < 0$);*
3. *The functional $L$ is bounded near $\mathcal{M}$ (there is some $\varepsilon_0 > 0$ and $C > 0$ such that $L(u) > -C$ if $d_2(u, \mathcal{M}) < \varepsilon_0$);*
4. *There is some $\varepsilon_1 > 0$ and some $\alpha > 0$ such that*

$$\frac{d}{dt}L(u(t)) = \langle DL(u(t)) \,,\, f(u(t), t)\rangle \leq \alpha L(u(t)), \tag{38}$$

*if $u(t)$ is a solution of (37), $d_2(u(t), \mathcal{M}) < \varepsilon_1$ and $L(u(t)) < 0$.*

*Then $\mathcal{M}$ is an unstable manifold. To be explicit, there is some $\varepsilon > 0$ such that for all $\delta > 0$ there is some solution $u(t)$ of (37) with $d_1(u(0), \mathcal{M}) < \delta$ and some $t_0 > 0$ such that $d_2(u(t_0), \mathcal{M}) > \varepsilon$.*

The proof of this lemma goes by contradiction. The assumption that $\mathcal{M}$ is stable and the integration of $L(u(t))$ for an initial condition $u(0)$ with $\widehat{L}_0(u(0)) < 0$ leads to the conclusion that $L(u(t))$ is unbounded below by using the fourth assumption. However, this contradicts the third assumption of the lemma. See [12] for details of this proof.

Comparing Lemma 4 with the previously mentioned Lyapunov type theorems for the instability of fixed points, we see that the conditions 1, 2 and 4 are very similar to the conditions in those theorems. Condition 3 is new and comes from the fact that the instability of a full (possibly unbounded) manifold is considered. Condition 3 implies that only a set of values of $L$ which is bounded below is relevant for this evolution. Thus even if $\mathcal{M}$ is unbounded,for a solution the functional $L$ can not become unbounded from below nearby the manifold.

Several refinements and variations of Lemma 4 are possible, see [12]. Especially if we consider a finite dimensional dynamical system, the last condition can be relaxed to $\langle DL(u) \,,\, f(u, t)\rangle < 0$.

In the Introduction and the examples in sections 2 and 4 we have seen that for a family of relative equilibria one can define a natural Lyapunov functional which satisfies the first condition of Lemma 4. The second condition states that the Lyapunov functional is indefinite and the third condition implies a kind of uniform continuity of the Lyapunov functional near $\mathcal{M}$. The last condition of Lemma 4 is not unexpected if one deals with a dissipative perturbation. Hence Lemma 4 suggests that often families of relative equilibria will be unstable under dissipation if the relative equilibria are not constraint extrema. In the next section this suggestion will be verified for the families $\mathcal{E}_k$, $k \geq 1$, in the Navier-Stokes equations.

## 6 Unstable families in Navier-Stokes equations

In this section we will use Lemma 4 to show that the families $\mathcal{E}_k$, $k \geq 1$, are unstable under Navier-Stokes equations. We have seen that the stability of those families under Euler equations is unknown, since the relative equilibria in those families are not constrained extrema.

Just as in the proof of the stability result for the family $\mathcal{E}_0$, a weighted Lyapunov functional

$$\widehat{L}_k(\omega) = \frac{L_k(\omega)}{H(\omega)} = \frac{W(\omega) - \lambda_k H(\omega)}{H(\omega)} \tag{39}$$

will be used to prove the instability of the other families. One can expect that a weighted functional is needed, since the zero state is a global attractor and the zero state is part of all families $\mathcal{E}_k$. So only instability in a cone-like neighbourhood can be expected, just as the stability result only holds in a cone-like neighbourhood.

The functional $\widehat{L}_k$ satisfies the following criteria.

1. At the family $\mathcal{E}_k$ the functional $\widehat{L}_k$ vanishes.
2. For any perturbation from the family $\mathcal{E}_k$ in the direction of $\mathcal{E}_0$, the functional $\widehat{L}_k$ is negative (for any $\eta > 0$ and $\overline{\omega} \in \mathcal{E}_k$, it holds $\widehat{L}_k(\overline{\omega} + \eta\Omega_0) = \frac{\eta^2(\lambda_0 - \lambda_k)}{\lambda_0 W(\overline{\omega}) + \eta^2 \lambda_k} < 0$).
3. $\widehat{L}_k$ is bounded below, since for any $\omega$ a Poincare inequality shows that $W(\omega) \geq \lambda_0 H(\omega)$, hence $\widehat{L}_k(\omega) \geq \lambda_0 - \lambda_k$.
4. For $\omega(t)$ near the family $\mathcal{E}_k$ it holds that if $\widehat{L}_k(\omega(t)) < 0$ then
$$\frac{d}{dt}\widehat{L}_k(\omega(t)) \leq \nu(\lambda_k - \lambda_{k-1})\widehat{L}_k(\omega(t)). \tag{40}$$
Here "near" means that
$$\min_{\overline{\omega} \in \mathcal{E}_k} \frac{\|\omega(t) - \overline{\omega}\|^2}{\|\nabla\psi(t)\|^2} < \frac{\lambda_0(\lambda_k - \lambda_{k-1})}{2(\lambda_k - \lambda_0)}, \tag{41}$$
with $\psi$ the stream function related to the vorticity $\omega$. Hence the neighbourhood is cone-like, see Figure 6.

To verify the last criterion one has to use appropriate Poincare inequalities, see [12].

With Lemma 4 we can draw the following conclusion.

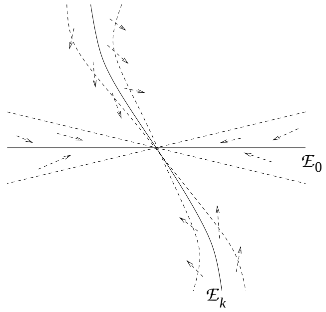


Figure 6. The family $\mathcal{E}_k$ is unstable in a cone-like neighbourhood.

**Theorem 5** *For all $k \geq 1$ the families $\mathcal{E}_k$ are unstable under Navier-Stokes equations. To be specific, there is some $\varepsilon > 0$ such that for all $\delta > 0$ and all $\widehat{\omega} \in \mathcal{E}_k$ there is some solution $\omega(t)$ of the Navier Stokes equation with $\|\omega(0) - \widehat{\omega}\| < \delta$ and some $t_0 > 0$ such that*

$$\|\Pi_k^{\perp}\omega(t_0)\| > \varepsilon\|\nabla\psi(t_0)\|, \tag{42}$$

*with $\psi$ the stream function related to $\omega$ and $\Pi_k$ the $L^2$-orthogonal projection onto $\mathcal{E}_k$. Again $\|\cdot\|$ is the $L^2$ norm on $\mathcal{D}$.*

## 7 Conclusion

The results in the previous sections suggest that under dissipation families of extremal relative equilibria are attractors and families of non-extremal relative equilibria are unstable. In order to prove such a statement, one would have to specify the term "dissipation". In contrary to the term "Hamiltonian", the term dissipation is not clearly defined. Sometimes dissipation is defined by using a metric and a dissipation functional, as opposed to a Hamiltonian system which is described by a symplectic form and a Hamiltonian functional. These two opposing effects are considered in [21,22,33], where the metric is such that the Hamiltonian and the momenta are conserved under the dissipation. Also in [25] the influence of this type of dissipation on Hamiltonian systems is studied. In this paper a dissipation of the form $P(u) = -D(u)\nabla H(u)$, with $D$ is a symmetric, positive semi-definite matrix, is considered. The zero state is a fixed point of this system. It is shown that if the condition $(D^2H(0)DP(0)u, u) < 0$ is satisfied for all (complex) eigenvectors $u$ of the linearised Hamiltonian system, then the zero state is linearly unstable if it is not an extremum of the Hamiltonian system.

Another problem, even with the definitions for dissipation as mentioned above, is that there are many possible ways to split a vector field in a Hamiltonian part and a dissipative part. An illustration of this phenomenon is given in the example of the spherical pendulum. The dissipation is this example is friction, however in the analysis of the pendulum, it turns out that it is more appropriate to split the friction term in a uniformly damping term and a Hamiltonian term. So this leads to the question whether there is an optional splitting. Such a splitting will probably depend both on the Hamiltonian and the dissipation. One suggestion for an "optimal" split in a the context of extremal relative equilibria with dissipation is that the split has to be such that the tangential dissipation coefficient, as defined in (6), is equal to the largest real part of the eigenvalues in the linearisation around the relative equilibrium.

As far as instability is concerned, the fourth condition in the general instability lemma (Lemma 4) gives a characterisation of the type of dissipations which will lead to instability. Preliminary work on the influence of friction on relative equilibria in simple mechanical systems suggest that the results of [5] can be extended and that one can conclude that non-extremal relative equilibria in simple mechanical systems become unstable if friction is added, also if the friction does not conserve the momentum map. An extra feature that is comes up in the analysis if the momentum map is not conserved is the curvature of the family of relative equilibria.

In this note, if the relative equilibria are related to symmetry, only dissipation which is equivariant under the symmetry group is considered. It is not immediately clear what the extra effects of a non-invariant dissipation would be. The Lyapunov functional as described before can still be used as distance functional. However, the estimates for the time behaviour will probably become harder since they will depend on the action of the group orbit as well. One might hope that for compact group actions one can get around this problem by using averaging techniques, [34] but it is not clear how sharp the estimates in stability case will be.

## References


1. R. Abraham and J.E. Marsden. *Foundations of Mechanics.* The Benjamin/Cummings Publ. Comp., 2nd edition (1978).
2. V.I. Arnold. *Doklady Mat. Nauk.* **162**, 773 (1966).
3. V.I. Arnold. *English Transl: Am. Math. Soc. Transl.* **19**, 267 (1969).
4. T. Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations.* Springer Verlag (1982).

5. A. Bloch, P.S. Krishnaprasad, J.E. Marsden, and T. Ratiu. *Ann. Inst. H. Poincaré, Anal. Nonl.* **11**, 37 (1994).
6. A. Bloch, P.S. Krishnaprasad, J.E. Marsden, and T. Ratiu. *Comm. Math. Phys.* **175**, 1 (1996).
7. S.J. Chern and J.E. Marsden. *Astr. Fluid. Dyn.* **51**, 1 (1990).
8. G. Derks and T.P. Valkering. *Jap. Journ. of Indus. and Appl. Math.* **9**, 141 (1992).
9. G. Derks, D. Lewis and T. Ratiu. *Nonlinearity* **8**, 1087 (1995).
10. G. Derks and E. van Groesen. *SIAM Journ. of Math. Anal.* **27**, 1424 (1996).
11. G. Derks and T. Ratiu. *Proc. Roy. Soc. Lond. A* **454**, 1407 (1998).
12. G. Derks and T. Ratiu. Preprint (1999).
13. T. Dombre, U. Frisch, J.M. Greene, M. Hénon, A. Mehr and A.M. Soward. *J. Fluid Mech.* **167**, 353 (1986).
14. C. Foias and J.C. Saut. Indiana Univ. Math. Jl **33**, 459 (1984).
15. N. Fenichel, N. Indiana Univ. Math. J. **21**, 193 (1971).
16. E.W.C. van Groesen. *Physica A* **148**, 312 (1988).
17. W. Hahn. *Stability of Motion.* Springer Verlag (1967).
18. E. Hairer and C. Lubich. *Applied Numerical Mathematics* **29**, 5 (1999).
19. M.W. Hirsch, C.C. Pugh and M. Shub. *Invariant manifolds.* Springer Verlag (1977).
20. D.D. Holm, J.E. Marsden, T. Ratiu, and A. Weinstein. *Physics Reports* **123**, 1 (1985).
21. A.N. Kaufman. *Physics Letters A* **100**, 419 (1984)
22. A.N. Kaufman. *Physics Letters A* **109**, 87 (1985)
23. P.S. Krishnaprasad and J.E. Marsden. *Arch. Rat. Mech. Anal.* **98**, 137 (1987).
24. N.R. Lebovitz and A. Neishtadt. *Stud. in Appl. Math.* **92**, 127 (1994).
25. J.M. Maddocks and M.L. Overton. *Comm. Pure Appl. Maths* **48**, 1 (1995).
26. I.G. Malkin. *Theory of Stability of Motion* (Transl). USA Atomic Energy Commission, (1952).
27. J.E. Marsden and A. Weinstein. *Reports on Mathematical Physics* **5**, 121 (1974).
28. J.E. Marsden. *Lectures on Mechanics.* Cambridge University Press (1992).
29. J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry*, 2nd ed. Springer Verlag (1999).
30. D.R. Merkin, *Introduction to the Theory of Stability.* Springer-Verlag (1997).

31. K.R. Meyer. *Symmetries and integrals in mechanics*, in "Dynamical Systems" (M.M. Peixoto, ed.), p. 259, Academic Press Inc. (1973).
32. J. Montaldi. *Nonlinearity* **10**, 449 (1997).
33. P.J. Morrison. *Physica D* **18**, 410 (1986).
34. J.A. Sanders and F. Verhulst. *Averaging methods in nonlinear dynamical systems.* Springer Verlag (1985).
35. J.C. Simo, D. Lewis, and J.E. Marsden. *Arch. Rat. Mech. Anal.* **115**, 15 (1991).
36. J.C. Simo, T.A. Posbergh, and J.E. Marsden. *Arch. Rat. Mech. Anal.* **115**, 60 (1991).
37. D. Stoffer. *J. Math. Anal. Appl.* **217**, 521 (1998).
38. V.I. Zubov. *Methods of A.M. Lyapunov and their applications.* P. Noordhoff (1964).