DIAGRAMMATIC TECHNIQUES IN PERTURBATIONS THEORY, AND APPLICATIONS

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Abstract: We want to discuss an approach to the study of perturbed systems in Classical Mechanics, based on methods characteristic of Quantum Field Theory (QFT). The starting point is to consider the perturbative expansion of the solution of the equations of motion: this leads to formal power series in the perturbative parameter, which were already considered and studied in the old papers by Newcomb, Lindstedt and Poincaré at the end of the last century. The novelty consists in applying to the study of such expansions techniques and tools which have been developed only recently (with respect to the quoted authors) in QFT, so in a very different field of Mathematical Physics. Applications and original results are briefly discussed.

1 Methods and Techniques

1.1 Introduction

The problems we shall consider will concern essentially finite dimensional Hamiltonian systems: in particular we shall consider perturbations of soluble systems and we shall be interested in the problem of persistence of invariant manifolds under (small) perturbations: KAM tori, low dimensional tori (with their stable and unstable manifolds), *etc.*

Anyway we shall study also area preserving maps, concentrating essentially on the standard map. This for two reasons: first there are a lot of still open problems about the standard map (which yet can be considered one of the simplest nontrivial systems); moreover the standard map, as we shall see, can be considered a paradigmatic example in which most of the interesting features of the methods (also for the continuous systems) appear without involving unessential technical intricacies.

In this section we shall discuss the techniques, dealing only with the standard map, essentially for simplicity of exposition. More precisely in subsection 1.2 we shall introduce the diagrammatic formalism and in subsection 1.3 we shall see how the QFT techniques can be applied to the problem. In subsections 1.4 and 1.5 (and in section 4) the more technical aspects are relegated, while the analysis of the previous subsections remains on a more qualitative level: if one wants just to have an idea of the proof without entering in details one can read only the subsections 1.2 and 1.3.

In section 2 we consider possible extensions of the formalism to more complicated situations. Endly in section 3 we shall briefly review some original results obtained with the described techniques.

1.2 Perturbative expansion

Let us consider for the moment only the standard map, defined by the sympletic map from the cylinder to itself

$$x' = x + y + \varepsilon \sin x, \qquad y' = y + \varepsilon \sin x, \tag{1}$$

where $(x, y) \in \mathbf{T} \times \mathbf{R}$, with $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$.

For $\varepsilon = 0$ the motion is trivial: one has a simple rotation $x' = x + 2\pi\omega$, while y is fixed to $y = 2\pi\omega$, $\omega \in [0, 1)$. For $\varepsilon \neq 0$ one can look for solutions which are conjugate to a trivial rotation of some other variable (KAM invariant curves), *i.e.* one can look for solutions of the form

$$x = \alpha + u(\alpha, \varepsilon), \qquad y = 2\pi\omega + v(\alpha, \varepsilon), \tag{2}$$

with $\alpha \to \alpha' = \alpha + 2\pi\omega$ and the functions u, v depending analytically on their arguments. The number ω will be called the *rotation number*.

If such solutions exist then it has to be possible to expand the functions u, v as Taylor series in ε and as Fourier series in α. So we are led to write, at least formally,

$$u(\alpha, \varepsilon) = \sum_{\nu \in \mathbf{Z}} \sum_{k=1}^{\infty} u_\nu^{(k)} \, e^{i\nu\alpha} \, \varepsilon^k, \qquad v(\alpha, \varepsilon) = \sum_{\nu \in \mathbf{Z}} \sum_{k=1}^{\infty} v_\nu^{(k)} \, e^{i\nu\alpha} \, \varepsilon^k. \tag{3}$$

The functions u, v are not independent from each other. One has $v(\alpha, \varepsilon) = u(\alpha, \varepsilon) - u(\alpha - 2\pi\omega, \varepsilon)$, as it is straightforward to check: simply note that $x' = x + y'$ by (1) and express x' and y' in terms of α through (2), using that $\alpha' = \alpha + 2\pi\omega$. So one obtains a closed equation for u:

$$Du(\alpha, \varepsilon) \equiv u(\alpha + 2\pi\omega, \varepsilon) + u(\alpha - 2\pi\omega, \varepsilon) - 2u(\alpha, \varepsilon) = -\varepsilon \partial \cos(\alpha + u(\alpha, \varepsilon)), \tag{4}$$

if ∂ denotes derivative with respect to the argument, so implying, by calling $[F(\alpha, \varepsilon)]_\nu^{(k)}$ the coefficient of the function $F(\alpha, \varepsilon)$ with Fourier label ν and Taylor label k,

$$\gamma(\omega\nu) u_\nu^{(k)} = \left[-\partial \cos(\alpha + u(\alpha, \varepsilon)) \right]_\nu^{(k-1)}, \qquad \gamma(\omega\nu) = 2 \left[\cos(2\pi\omega\nu) - 1 \right]. \tag{5}$$

Note that, for $\omega\nu$ small (mod. 1), $\gamma(\omega\nu) \sim \|\omega\nu\|^2$, if $\|\cdot\|$ denotes the distance on the torus $\mathbf{T}$.

By explicitly writing $\cos\alpha = \sum_{\mu_0=\pm 1} 2^{-1} e^{i\mu_0\alpha}$ and Taylor expanding $\partial\cos(\alpha+u)$ in u around $u=0$, one finds, from (5),

$$\begin{aligned} u_\nu^{(k)} &= \frac{1}{\gamma(\omega\nu)} \sum_{m=0}^{\infty} \sum_{\substack{\mu_0+\nu_1+\ldots+\nu_m=\nu \\ k_1+\ldots+k_m=k-1}} \left[\frac{-(i\mu_0)^m}{m!2}\right] u_{\nu_1}^{(k_1)} \ldots u_{\nu_m}^{(k_m)}, \qquad k>1, \\ u_\nu^{(1)} &= -\frac{i\nu}{2\gamma(\omega\nu)}. \end{aligned} \tag{6}$$

It is easy to check that in $u_\nu^{(k)}$ one has $0<|\nu|\le k$. Then (6) can be iterated by taking into account that $k_j<k$ for any $j=1,\ldots,m$, so allowing us to express the coefficients $u_\nu^{(k)}$ in terms of *trees*.

A tree consists in a set of lines connecting a partially ordered set of points; denoting with $\le$ the ordering relation we say that a point w is lower than a point v if $w\le v$. A line connecting a point w to a point v is said to exit w and enter v. All the points except the leftmost one are called nodes, with the lower nodes to the right and the higher to the left. Each line carries an arrow pointing toward the leftmost point of the tree which is called the *root* of the tree (and which is not a node); the line entering the root is called the *root line*. The number of nodes in ϑ is the *order* of the tree ϑ. Details on trees can be found in any book on such a subject [23].

To any node v a *mode* label $\mu_v \in \{\pm 1\}$ and to each line λ a *momentum* label $\nu_\lambda \in \mathbf{Z}$ are associated: we say the the momentum ν_λ flows through the line λ. For each node there is a *conservation law* stating that if μ_v is the mode label of the node v, ν is the momentum of the line exiting the node and $\nu_1,\ldots,\nu_{m_v}$ are the momenta of the m_v lines entering the node, then $\nu=\mu_v+\nu_1+\ldots+\nu_{m_v}$.

Let us denote by $\mathcal{T}_{\nu,k}$ the set of all trees with k nodes and with momentum ν flowing through the root line. The number of elements in $\mathcal{T}_{\nu,k}$ is bounded by $2^k 2^{2k}$.

Then the iteration of (6) and the definition of trees lead to the following expression:

$$u_\nu^{(k)} = \sum_{\vartheta\in\mathcal{T}_{\nu,k}} \mathrm{Val}(\vartheta), \qquad \mathrm{Val}(\vartheta) = \left[\prod_{v\in\vartheta} f_v\right]\left[\prod_{\lambda\in\vartheta} g_\lambda\right], \tag{7}$$

with

$$f_v = -\frac{(i\mu_v)^{m_v}}{m_v!2}, \qquad g_\lambda = \frac{1}{\gamma(\omega\nu_\lambda)}, \tag{8}$$

where the factors g_λ are called *propagators* (or *small divisors*). The conservation law and the fact that for each node v one has $\mu_v=\pm 1$ yield the already remarked property that $|\nu|\le k$. Then to prove the convergence of the series (3), it is enough to show that a bound $|u_\nu^{(k)}|\le C^k$ holds for some constant C: this gives an estimate $\epsilon_0 = C^{-1}$ for the radius of convergence of the series.

1.3 Quantum Field Theory formalism and renormalization

Until now nothing new has been really obtained. We simply developed a tree expansion for the formal solution of (4), which of course can be attempted easily for any function recursively defined as the function u is, so looking for a solution of the equations of motion formally well defined to all orders.

Then, in order to check if not only the series in ε defining u is formally defined but also it converges, one needs some extra work. The real problems arise from the propagators g_λ, as the factors f_v in (8) are trivially bounded by 1.

First of all note that some condition has to be imposed on the rotation number ω, for instance a Diophantine condition

$$\|\omega\nu\| > C_0|\nu|^{-\tau}, \qquad 0 \neq \nu \in \mathbf{Z}, \tag{9}$$

where $C_0 > 0$ and $\tau > 1$, or a (weaker) Bryuno condition

$$B(\omega) \equiv \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty, \tag{10}$$

where $\{q_n\}_{n=0}^{\infty}$ are the denominators of the convergents of the continuous fraction expansion for ω.

Then let us introduce a label characterizing the size of the propagators. More formally we introduce a C^∞ partition of unity

$$1 = \sum_{n=0}^{\infty} \chi_n(\|\omega\nu\|), \tag{11}$$

where, given a strictly decreasing sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers converging to zero, for any $n \geq 1$, χ_n is a C^∞ function with compact support $[a_{n+1}/N_1, a_n/N_2]$, where $N_1 > N_2$ are integers numbers, while χ_0 is a C^∞ function with support $[a_0/N_1, \infty)$ (regularized step function).

For instance, under the Diophantine condition (9), one can take

$$N_1 = 2, \qquad N_2 = 1, \qquad a_n = C_0 2^{-n}. \tag{12}$$

Then, for each propagator g_λ, we introduce the *multiscale decomposition*

$$g_\lambda = \frac{1}{\gamma(\omega\nu_\lambda)} = \sum_{n=0}^{\infty} \frac{\chi_n(\|\omega\nu_\lambda\|)}{\gamma(\omega\nu_\lambda)} \equiv \sum_{n=0}^{\infty} g_\lambda^{(n)}, \tag{13}$$

so defining $g_\lambda^{(n)}$ a *propagator on scale* n. Note that, fixed a momentum $\nu_\lambda = \nu$, only a finite number of scales is compatible with ν, *i.e.* only for a finite number s of scales n one has $\chi_n(\|\omega\nu\|) \neq 0$; for instance with the choice (12) only two scales are compatible with a fixed momentum ν, hence $s = 2$. If a propagator is on scale n and $\chi_n(\|\omega\nu\|) \neq 0$, then $|g_\lambda^{(n)}| < G_0 a_{n+1}^{-2}$ for some constant G_0.

So in a given tree ϑ we have an extra label n_λ for each line $\lambda \in \vartheta$. Then (7) has to be replaced with

$$u_\nu^{(k)} = \sum_{\vartheta \in \mathcal{T}^*_{\nu,k}} \mathrm{Val}(\vartheta), \qquad \mathrm{Val}(\vartheta) = \left[\prod_{v \in \vartheta} f_v\right] \left[\prod_{\lambda \in \vartheta} g_\lambda^{(n_\lambda)}\right], \tag{14}$$

where $\mathcal{T}^*_{\nu,k}$ differs from the set $\mathcal{T}_{\nu,k}$ inasmuch the lines have also a scale label: the number of elements in $\mathcal{T}^*_{\nu,k}$ is bounded by $s^k 2^k 2^{2k}$.

The scale labels allow us to introduce [11] the notion of *cluster* T on scale n as a maximal set of lines on scales $\le n$ such that at least one line is on scale n; the nodes connected by the lines in T are said to belong to T. A picture could be very useful at this point [22].

Among the clusters we define a *resonance* as a cluster such that there is only one entering line λ_1 and only one exiting line λ_2, carrying the same momentum $\nu_{\lambda_1} = \nu_{\lambda_2}$: this implies $\sum_{v \in V} \mu_v = 0$. We call *resonant line* a line entering a resonance.

As we shall see later (see subsection 1.5), the definition of resonance has to be slightly modified in order to work. Nevertheless for a heuristic discussion, as the one in the present subsection is, we prefer to ignore such a problem.

Let us define

$$\begin{aligned} N_n &= \text{number of lines on scale } n, \\ N_n^R &= \text{number of resonant lines on scale } n, \\ N_n^* &= \text{number of nonresonant lines on scale } n, \\ p_n &= \text{number of clusters on scale } n, \end{aligned} \tag{15}$$

so that $N_n = N_n^* + N_n^R$. Then one has,

$$\prod_{\lambda \in \vartheta} \left| g_\lambda^{(n_\lambda)} \right| \le G_0^k \prod_{n=0}^{\infty} a_{n+1}^{-2N_n}, \tag{16}$$

which, in the case (12), gives (with $\tilde{G}_0 = 2^2 G_0 / C_0^2$)

$$\prod_{\lambda \in \vartheta} \left| g_\lambda^{(n_\lambda)} \right| \le \tilde{G}_0^k \prod_{n=0}^{\infty} 2^{2nN_n} = \tilde{G}_0^k \left[\prod_{n=0}^{\infty} 2^{2nN_n^*}\right] \left[\prod_{n=0}^{\infty} 2^{2nN_n^R}\right]. \tag{17}$$

The reason why the writing (17) is useful relies on the fact that, given a tree $\vartheta \in \mathcal{T}^*_{\nu,k}$, then the following result (*Siegel-Bryuno's lemma*) holds:

$$N_n^* \le Ak2^{-n/\tau}, \qquad p_n \le Ak2^{-n/\tau}, \tag{18}$$

for some constant A. The proof is a minor adaptation from the continuous time case [11] and it is presented for completeness in subsection 4.1 (and it gives $A = 2^{2/\tau}2$).

If the resonant lines could be neglected, then the bound

$$\prod_{n=0}^{\infty} 2^{2nN_n^*} \le \exp\left[2Ak\log 2\sum_{n=0}^{\infty} n2^{-n/\tau}\right] \equiv A_0^k, \tag{19}$$

inserted into (17), would imply, to order k, a coefficient $u_\nu^{(k)}$ bounded by a constant to the power k, so yielding summability on k.

However there are the resonant lines and we have to deal with them. Note that this is not a problem of improving the bound given by Siegel-Bryuno's lemma: in fact one can construct explicit examples [10] of trees whose value is of order of $k!^a$, for some constant $a > 0$. The only hope which we are left with is that there can be some cancellation mechanisms operating to all perturbative orders between huge tree values in such a way that, even if there are single trees admitting a bad bound (factorial to some power), nevertheless the sum of the values of all trees of order k still admits a good bound (a constant to the power k).

Then let us consider a resonance V. Define the *resonance factor*

$$\mathcal{V}_V(\omega\nu) = \left[\prod_{v\in V} f_v\right]\left[\prod_{\lambda\in V} g_\lambda\right], \tag{20}$$

which will be considered as a function of the quantity $\omega\nu$, if ν is the momentum flowing through the resonant line. Set

$$\begin{aligned}\mathcal{V}_V(\omega\nu) &= [\mathcal{V}_V(0) + \partial\mathcal{V}_V(0)\,\omega\nu] + \left\{\int_0^1 dt_V\,(1-t_V)\partial^2\mathcal{V}_V(t_V\omega\nu)\,(\omega\nu)^2\right\} \\ &= [\mathcal{L}\mathcal{V}_V(\omega\nu)] + \{\mathcal{R}\mathcal{V}_V(\omega\nu)\},\end{aligned} \tag{21}$$

where $\mathcal{L}\mathcal{V}_V(\omega\nu)$ is called the *localized resonance factor*, while $\mathcal{R}\mathcal{V}_V(\omega\nu)$ is called the *renormalized resonance factor*. In (21) the quantity $\omega\nu$ is defined modulo 1 and ∂ denotes the derivative with respect to the argument; $\mathcal{L}$ and $\mathcal{R}$ are called, respectively, the *localization operator* and the *renormalization operator*.

Again, if one could neglect the localized resonance factor, no problem should arise. Let us suppose that we are dealing only with trees without localized resonance factors: let us call *renormalized trees* such trees. The factor $(\omega\nu)^2$ appearing in the renormalized resonance factor simply deletes the propagator corresponding to the resonant line (see (8) and the comment after (5)), while the double derivative acting on $\mathcal{V}_V$ at most generates an extra factor $G_1 2^{2n_V}$, if n_V is the scale of the resonance as a cluster and G_1 is a suitable constant: this follows from the remark that the derivatives act on the propagators of the lines internal to V and all such propagators are on scale $\le n_V$. One has at most one of such extra factors for each resonance, so that, in conclusion, we are able to get rid of the factors $2^{2nN_n^R}$ in (17), provided a factor 2^{2n_V} is introduced for each resonance. But

$$\prod_{V\in\vartheta} 2^{2n_V} \le \prod_{n=0}^{\infty} 2^{2np_n}, \tag{22}$$

so that Siegel-Bryuno's lemma can be used once more to deduce that a bound C^k is obtained on $|u_\nu^{(k)}|$ for some constant C.

At this point problems of two kinds still remain. There are technical problems in order to make rigorous the above (still) heuristic discussion about the renormalized trees and there are conceptual problems concerning the localized resonance factors. The first ones can be dealt with in a standard way (at least "standard" in the framework of QFT) and are not of particular interest here: we shall briefly discuss them in subsection 1.4 below.

The latter, on the contrary, are the only real problems and there is no "general" device in order to solve them: anyway one can prove, exactly as in the case of continuous-time systems [21], that for the standard map there are, as we anticipated, cancellations between tree values. A more technical discussion is deferred to subsection 1.5 below. Here we only remark that the cancellations are related to the symplectic structure of the equations of motion, hence to an underlying symmetry of the problem which has not really been used to full extent until now. In a recent paper [18], the analogy with QFT, already used in previous works [14], was stressed by showing that the cancellations can be interpreted as Ward identities of a suitable QFT model.

1.4 Renormalized trees

Let us continue for the moment to suppose that no contribution arises from the localized resonance factors; this will be proven in next subsection 1.5.

The analysis of the previous section is complete as far as only one resonance intervenes in the discussion. If there are more resonances, all disjoint, the analysis still applies as there is no interference between the involved resonance factors.

On the contrary problems arise if there are resonances contained inside other resonances: in fact the operator $\mathcal{R}$, when acting on a resonance V, requires to compute the second derivative of the resonance factor $\mathcal{V}_V$ and the derivative can act on a propagator corresponding to a line λ inside some resonance $V' \subset V$; then when also the operator $\mathcal{R}$ on V' is taken into account one could derive twice more the propagator $g_\lambda^{(n_\lambda)}$. And so on: if there is a chain of resonances contained into each other, one can derive some propagators up to twice the maximal number of resonances that can be present in a tree, number which can be bounded by k, if k is the number of nodes in ϑ. Moreover the derivatives can act on the compact support functions appearing in the propagators and no control exists on the derivatives of such functions: the pth derivative of χ_n grows worse than $p!$.

But the problem is not real and the difficulty can be easily solved. Suppose to have a resonance V_2 inside a resonance V_1. In order to apply the results in the previous section we would want, without deriving more than twice each propagator, a gain $2^{2n_{V_1}-2n_1}\, 2^{2n_{V_2}-2n_2}$, if n_1 and n_2 are the scales of the resonant lines of V_1 and V_2, respectively.

Consider the $\mathcal{R}$ operator applied to $\mathcal{V}_{V_1}$: the two derivatives of $\mathcal{R}$ can act either both on lines external to V_2 or one on a line internal and one on a line external to V_2 or both on lines internal to V_2. In the first case, when considering $\mathcal{R}\mathcal{V}_{V_2}$ no more than two derivatives can act on the lines internal to V_2 (the two of $\mathcal{R}$ applied to V_2 itself). In the other two cases we can use the fact that, for any resonance V, the renormalized resonance factor can be written both as in (21) and as $\mathcal{R}\mathcal{V}_V(\omega\nu) = \mathcal{V}_V(\omega\nu) - \mathcal{L}\mathcal{V}_V(\omega\nu)$. Then if only one derivative acts on $\mathcal{V}_{V_2}$ and ν, ν' are the momenta flowing through the lines entering V_1, V_2 (hence $\omega\nu' = \omega\nu'_0 + t_V\omega\nu$, for some $\nu'_0 \in \mathbf{Z}$), we can write

$$\begin{aligned} \partial\mathcal{R}\mathcal{V}_{V_2}(\omega\nu') &= \partial\left(\mathcal{V}_{V_2}(\omega\nu') - \mathcal{L}\mathcal{V}_{V_2}(\omega\nu')\right) = \left[\partial'\mathcal{V}_{V_2}(\omega\nu') - \partial'\mathcal{V}_{V_2}(0)\right] t_V \\ &= \int_0^1 dt_{V'}\, \partial'^2\mathcal{V}_{V_2}(t_{V'}\omega\nu')\,(\omega\nu')\, t_V, \end{aligned} \tag{23}$$

with $\partial = \partial/\partial(\omega\nu)$ and $\partial' = \partial/\partial(\omega\nu')$, while, if two derivatives act on $\mathcal{V}_{V_2}$, we can write

$$\begin{aligned} \partial^2\mathcal{R}\mathcal{V}_{V_2}(\omega\nu') &= \partial^2\left(\mathcal{V}_{V_2}(\omega\nu') - \mathcal{L}\mathcal{V}_{V_2}(\omega\nu')\right) = \partial^2\mathcal{V}_2(\omega\nu') \\ &= \partial'^2\mathcal{V}_2(\omega\nu')\, t_V^2, \end{aligned} \tag{24}$$

so that one sees that two derivatives act on $\mathcal{V}_{V_2}$ in any case.

If both derivatives due to the renormalization of V_1 act on lines inside V_2 we obtain a gain $2^{2n_{V_2}-2n_1}$ and each line is derived at most twice: by using the inclusion relation between clusters, we can bound $2^{2n_{V_2}-2n_1} \le 2^{2n_{V_2}-2n_2}\, 2^{2n_{V_1}-2n_1}$. An analogous gain is obtained for the left out cases.

Note that if the resonance V_2 contains a third resonance V_3 we can proceed exactly in the same way. Therefore we can consider an arbitrary number of resonances contained into each other and we can apply the same argument: at the end (22) applies.

1.5 Localized resonance factors: cancellations

The cancellations have to been looked for between trees which belong to suitable *resummation families*, defined as follows

Consider a tree ϑ with a resonance V. Define $\mathcal{F}_V(\vartheta)$ as the family of all trees obtained from ϑ by the action of a group $\mathcal{P}_V$ of transformations generated by the two following elementary operatios: (1) detach the (only) line entering V and reattach it to any node inside V; (2) flip the mode labels of all nodes inside V simultaneously.

Each tree ϑ' in $\mathcal{F}_V(\vartheta)$ inherits the scale labels of ϑ (up to such a difference, the value of ϑ' is given as in (14)). Then it can happen that its value is vanishing: in fact the shift of the line entering the resonance can lead to a change of some propagators, so that the compatibility condition remarked after (13) can fail to be fulfilled. This means that for such a tree ϑ' the resonance factor (21) is vanishing: nevertheless we can always decompose it as sum of a localized part plus a renormalized part. If the localized part of the resonance factor for ϑ was nonvanishing then also the localized part of the resonance factor for ϑ' is nonvanishing. This follows from the fact that for any tree in $\mathcal{F}_V(\vartheta)$ the propagators appearing in the localized resonance factor are the same: simply note that the momenta flowing through the lines inside V, as far as the localized resonance factor is concerned, are obtained by setting equal to zero the momentum flowing through the entering line, so it does not matter which node the entering line is attached to.

Then consider the sum of the values of all trees inside the same family $\mathcal{F}_V(\vartheta)$, retaining only the localized part of the resonance factor. Since for all trees in $\mathcal{F}_V(\vartheta)$, the propagators inside V do not depend on the particular tree and the structure of the tree outside the resonance is the same, the only differences between the values of such trees are due to the factors f_v, $v \in V$. It is not difficult to check that, to zeroth order in $\omega\nu$, the following relation holds:

$$\sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \prod_{v \in V} f_v = \mathcal{A}(\vartheta) \sum_{v \in V} \mu_v, \tag{25}$$

for some function $\mathcal{A}(\vartheta)$ depending only on ϑ. To check (25) simply use the definition of f_v in (8); the only delicate point concerns the combinatorial factors, but this can be easily solved and finally leads to (25). By definition of resonance the quantity in (25) is vanishing.

To deduce the cancellation also to first order note that the first order contribution to the localized resonance factor is odd by change of signs of all mode labels μ_v, $v \in \vartheta$, so that the second elementary operation generating the group $\mathcal{P}_V$ leads to a cancellation also to first order.

We can conclude from the above discussion that

$$\sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \mathcal{L}\mathcal{V}_V(\vartheta) = 0, \tag{26}$$

if V is a resonance in ϑ.

Of course there can be more resonances, but as there can not be overlapping between resonances (two resonances either are disjoint or are strictly contained into each other), the cancellation mechanism described above works for all resonances separately. The conclusion is therefore that, as far as only the localized part of the resonances is retained for all resonances inside a tree ϑ, the *resummation procedure* consisting in summing up the values of all trees in $\mathcal{T}^*_{\nu,k}$ gives a vanishing contribution. Therefore we can confine ourselves to the renormalized resonance factors and apply to them the analysis of the previous subsection.

There is only a *caveat*: we noted that the resummation procedure can lead to the introduction of trees which have scale labels and momenta not compatible with each other. So we have to control both that such new trees are not too much and that for such trees the implications of Siegel-Bryuno's lemma are still valid.

Let us be more precise on such difficulties. We know that the number of elements in $\mathcal{T}^*_{\nu,k}$ is bounded by 2^{4k} for $s = 2$ (see subsection 1.3). So we want that the number of elements added by the resummation procedure leads to an overall number of elements which is still bounded by a constant to the power k.

Moreover Siegel-Bryuno's lemma is proven by using that if a line has momentum ν then it has to be on a scale n such that $\chi_n(\|\omega\nu\|) \neq 0$.

Both problems can be solved modifying the definition of resonance by requiring the extra property, besides the one already stated in subsection 1.3, that the scale of the resonance V has to be $n \geq n_V + 5$, if n is the scale of the line entering V.

Consider a resonance V contained inside other resonances. The shifts of all lines entering such resonances produce subsequent changes of the momentum $\tilde{\nu}_\lambda$ of any line $\lambda \in V$ into a new value ν_λ. Even if summed together, they can not shift $\omega\tilde{\nu}_\lambda$ more than by $C_0 2^{-(n-2)}$ (such a property can be easily proved by induction): at worst, as $n_\lambda \leq n_V \leq n-5$ for any $\lambda \in V$,

$$\begin{aligned} 2^{-(n_\lambda+3)} & \leq 2^{-(n_\lambda+2)} - 2^{-(n-2)} \\ & \leq C_0^{-1}\|\omega\nu_\lambda\| \leq 2^{-n_\lambda} + 2^{-(n-2)} \leq 2^{-(n_\lambda-1)}. \end{aligned} \tag{27}$$

Then if the condition $n \geq n_V + 5$ is imposed, although it can happen that in some $\vartheta' \in \mathcal{F}_V(\vartheta)$ and for some line λ internal to V one has $\chi_n(\|\omega\nu_\lambda\|) = 0$, if n is the scale inherited from ϑ, nevertheless the scale n' compatible with ν_λ (that is the scale n' for which $\chi_{n'}(\|\omega\nu_\lambda\|) \neq 0$) is such that $|n-n'| \leq 1$.

Therefore, as the scale of a line can change at most by 1, the number of extra trees generated by the resummation procedure is bounded by 3^k times the number of elements in $\mathcal{T}^*_{\nu,k}$. Moreover Siegel-Bryuno's lemma still applies as a simple glance at its proof in subsection 4.1 shows. The only difference is that if a line λ is on scale n then $\|\omega\nu_\lambda\| \leq C_0 2^{-(n_\lambda-1)}$ instead of $\|\omega\nu_\lambda\| \leq C_0 2^{-n_\lambda}$; see end comments in subsection 4.1.

Of course with the new definition of resonance no gain is obtained when $n < n_V + 5$; however in such a case one has $2^{2n_V-2n} \leq 2^8$, so that we can replace 2^{2n} with 2^{2n_V} provided an extra factor 2^8 is introduced into the bounds.

2 Generalizations and extensions

2.1 Other maps

The extension of the formalism to area preserving maps, from the cylynder to itself, more general than the standard map is obvious. In particular, and in some sense by performing a step backwards with respect to the results in the previous section, the semistandard map

$$x' = x + y + \varepsilon \exp ix, \qquad y' = y + \varepsilon \exp ix, \tag{28}$$

can be considered. All the results valid for the standard map hold also for the semistandard map (and the corresponding proof becomes easier as no renormalization is needed).

2.2 Invariant tori for Hamiltonian flows

We can also consider continuous-time systems described by Hamiltonians of the form

$$\mathcal{H}(\underline{\alpha},\underline{A}) = \mathcal{H}_0(\underline{A}) + \varepsilon\, f(\underline{\alpha},\underline{A}), \tag{29}$$

where $(\underline{\alpha},\underline{A}) \in \mathbf{T}^\ell \times \mathbf{R}^\ell$ are conjugate variables (*action–angle variables*), ε is the *strenght* of the interaction and $\mathcal{H}_0$ and f are real analytic functions, $\mathcal{H}_0$ depending only on the actions and f periodic in the angles variables. In particular (for simplicity) we can consider Hamiltonians in which $\mathcal{H}_0$ is quadratic and f is an even trigonometric polynomial depending only on the angles, *i.e.*

$$\mathcal{H}_0(\underline{A}) = \frac{1}{2} J^{-1}\underline{A}\cdot\underline{A}, \qquad f(\underline{\alpha},\underline{A}) \equiv f(\underline{\alpha}) = \sum_{\substack{\underline{\nu}\in Z^\ell \\ |\underline{\nu}|\le N}} f_{\underline{\nu}}\, e^{i\underline{\nu}\cdot\underline{\alpha}}, \; f_{\underline{\nu}} = f_{-\underline{\nu}}, \tag{30}$$

which is sometimes [11] called *Thirring model* [26], and look for solutions conjugate to a trivial rotation (KAM invariant tori), *i.e.* of the form

$$\underline{\alpha}(t) = \underline{\alpha}_0 + 2\pi\underline{\omega}t + \underline{h}(2\pi\underline{\omega}t,\varepsilon), \qquad \underline{A}(t) = \underline{A}_0 + \underline{H}(2\pi\underline{\omega}t,\varepsilon), \tag{31}$$

with $\underline{h},\underline{H}$ analytic in their arguments, for ε small enough and $2\pi\underline{\omega}t \in \mathbf{T}^\ell$, and $J^{-1}\underline{A}_0 = 2\pi\underline{\omega}$, if $\underline{\omega}$ is some Diophantine vector, *i.e.*

$$|\underline{\omega}\cdot\underline{\nu}| > C_0\, |\underline{\nu}|^{-\tau}, \qquad 0 \neq \underline{\nu} \in \mathbf{Z}^\ell. \tag{32}$$

In (30) $\cdot$ denotes the inner product in $\mathbf{R}^\ell$ and J is a $\ell\times\ell$ positive definite matrix (no anisochrony condition is required on J: this is a remarkable property [11] due to the action-independency of the perturbation); in (32) we set $|\underline{\nu}| = |\nu_1| + \ldots + |\nu_\ell|$.

As for the standard map one finds that there is a relation between $\underline{h}$ and $\underline{H}$ (simply $\underline{H}$ is the time derivative of $\underline{h}$), so that a closed equation is found for $\underline{h}$:

$$(2\pi i\underline{\omega}\cdot\underline{\nu})^2\, \underline{h}_{\underline{\nu}}^{(k)} = \left[-\underline{\partial} f(\underline{\alpha}(t))\right]_{\underline{\nu}}^{(k)}, \tag{33}$$

where $\underline{\partial}$ denotes derivative with respect to the argument and $[\underline{F}(\underline{\alpha},\varepsilon)]_{\underline{\nu}}^{(k)}$ denotes the coefficient of the function $\underline{F}(\underline{\alpha},\varepsilon)$ with Fourier label $\underline{\nu}$ and Taylor label k. A tree representation – similar to the one envisaged in section 1 – can be developed also for the coefficients $\underline{h}_{\underline{\nu}}^{(k)}$ and the discussion of the previous section can be extended to such a case.

2.3 Low-dimensional tori and hiskers for Hamiltonian flows

One can also consider Hamiltonian systems of the form

$$\mathcal{H}(\underline{\alpha},\varphi,\underline{A},I) = \underline{\omega}\cdot\underline{A} + \frac{1}{2}J^{-1}\underline{A}\cdot\underline{A} + \frac{I^2}{2J_0} + g^2\,(\cos\varphi - 1) + \varepsilon\, f(\underline{\alpha},\varphi), \tag{34}$$

where $X = (\varphi,\underline{\alpha},I,\underline{A}) \in \mathbf{T}\times\mathbf{T}^\ell\times\mathbf{R}\times\mathbf{R}^\ell$ are canonically conjugate variables, g, J_0 are positive constants, J is a positive definite matrix and the perturbation

$f(\underline{\alpha}, \varphi)$ is an even trigonometric polynomial of degree N in $\underline{\alpha}$ and of degree N_0 in φ.

The unperturbed part in (34) describes a set of ℓ rotators and a free pendulum and the interaction is again an even trigonometric polynomial in (all) the angles.

For $\varepsilon = 0$ the systems admit 1-codimensional tori, corresponding to the hyperbolic fixed point of the pendulum, with their (degenerate) stable and unstable manifolds. For $\varepsilon \neq 0$ one can study the persistence of such low dimensiomal tori and their manifolds (called *whiskers*, so that the tori are usually denoted as *whiskered tori*): in general the stable and the unstable manifolds of such tori are no more degenerate, but intersect each other.

Then it is interesting to see if the angles between the two manifolds are nonvanishing [15], as such a phenomenon is related [1] to the possibility of having Arnol'd diffusion in phase space.

One looks for solutions which can be parameterized as $X = X(\underline{\omega}t, e^{-\sigma g t})$, with $\sigma = \text{sign}(t)$, by imposing that, for $t \to \infty$, they are asymptotic to invariant tori with rotation vector $\underline{\omega}$. The initial data are chosen as $\underline{\alpha} = \underline{0}$ and $\varphi = 0$ (and the action variables are consequently fixed); the reason why one chooses such initial data is that in correspondence of them, by parity arguments, one can deduce the existence of homoclinic intersection points. As in the case of KAM invariant tori one obtains [12] an expansion in terms of trees for motions occurring along the stable and the unstable manifolds of any persisting invariant tori, starting from the homoclinic intersection point.

3 Results

3.1 First steps

The renewed interest in the direct analysis of the perturbative series in Classical Mechanics, known as *Lindstedt series*, stemmed from a work (on the continuous-time systems) by Eliasson [10], appeared in 1988 as a preprint and only recently published. The ideas of that paper have then inspired other works by Gallavotti [11] and by Chierchia and Falcolini [8]; note that the somewhat stronger condition on the rotation vectors considered by Gallavotti was later removed [13]. In the form presented here they appeared in subsequent papers first for the Thirring model [21], then for the general case [22]. A proof of existence of the whiskered tori for the systems (34) can be given within the same framework [19], by using the formalism introduced by Gallavotti [12]. A variation of the proof [20] pushes forward the analogy with QFT through the introduction, as a useful device in order to attack the problem, of suitable *counterterms* in the Hamiltonian: the original Hamiltonian is then recovered by resolving an implicit theorem problem (consisting in inverting a close to identity function).

All such papers, anyway, simply present new proofs (to be added to the countless already existing in literature) of KAM theorem for maximal and low dimensional tori. Though we can remark that, in general, better bounds are obtained in this way on the radius of convergence in the perturbation parameter of the invariant tori. Nevertheless the real interest in such techniques relies on the possibility to obtain any original results. Some of them are discussed in the forthcoming subsections.

3.2 Differentiable perturbations

Let us consider Hamiltonians of the form (30), with $f(\underline{\alpha})$ only differentiable in $\underline{\alpha}$. Moser's theorem (*i.e.* Moser's version of KAM theorem) implies the existence of invariant tori in such a case. Define the class $\hat{C}^p(\mathbf{T}^\ell)$ of functions $f \in C^p(\mathbf{T}^\ell)$ which can be written as

$$f(\underline{\alpha}) = \sum_{\underline{\nu} \in \mathbf{Z}} f_{\underline{\nu}} e^{i\underline{\nu}\cdot\underline{\alpha}}, \tag{35}$$

with $f_{\underline{0}} = 0$ and, for $\underline{\nu} \neq \underline{0}$,

$$f_{\underline{\nu}} = \sum_{n \geq p+\ell}^{n_0} \frac{c_n + d_n(-1)^{|\underline{\nu}|}}{\sqrt{\underline{\nu}\cdot\underline{\nu}}}, \tag{36}$$

for some integer n_0. If the potential f belongs to the class $\hat{C}^p(\mathbf{T}^\ell)$, with $p > p_0 = 3\tau + 6$, then one can prove [6] that the invariant tori are *analytic* in the perturbation parameter ε. This is a nontrivial result, because if one tries to apply perturbation theory one has to face immediately the problem of deriving an arbitrary number of times a function differentiable only p times; anyway there are new cancellations (completely different from the ones briefly described in subsection 1.5) assuring the convergence of the formal series.

The class of functions to which the result applies can be enlarged up to cover a set dense [6] in $C^p(\mathbf{T}^\ell)$. The same algorithm can be applied to similar problems [7].

3.3 Radius of convergence of the standard map (I)

For the standard map one can studies the optimality of the dependence of the radius of convergence $\rho(\omega)$ on the rotation number ω. The discussion in subsection 1.3 can be adapted to the Bryuno condition (10); for instance one can choose, instead of (12), $a_n = q_n^{-1}$. Then the bound $\rho(\omega) > D_1 e^{-8B(\omega)}$ can be easily obtained (see subsection 4.2). The bound so obtained is not optimal: a more careful analysis [3], leading also to a new (very involved) definition of resonance, shows that a bound $\rho(\omega) > D_1 e^{-2B(\omega)}$ can be obtained. Such a bound is optimal: in fact a result by Davie [9], shows that an analogous bound holds also from above, so that one can conclude that $|\log\rho(\omega) + 2B(\omega)| < D_0$, for some constant D_0, so proving a conjecture by Marmi and Stark [24], based on Yoccoz' work [27] on the Siegel problem and numerical simulations.

3.4 Radius of convergence of the standard map (II)

Always for the standard map one can consider the perturbative series defining an invariant circle with rotation number $\omega = p/q + i\eta$, with p, q relatively prime integers and η real, in the limit $\eta \to 0$ (or more generally for ω tending to a rational value non-tangentially to the real axis). On the basis of numerical simulations the following result was conjectured [4]: by rescaling by $(2\pi\eta)^{2/q}$ the perturbation parameter and considering the limit

$$\bar{u}(\alpha, \varepsilon) = \lim_{\eta \to 0} u(\alpha, (2\pi\eta)^{2/q}\varepsilon), \tag{37}$$

then such a limit exists, is analytic in ε^q and α, is $2\pi/q$-periodic in α and solves the differential equation

$$\frac{d^2\bar{u}}{d\alpha^2} = C_{p/q}\varepsilon^q \sin\left[q\left(\alpha + \bar{u}\right)\right], \qquad \bar{u}(0, \varepsilon) = \bar{u}(2\pi, \varepsilon) = 0, \tag{38}$$

for some constant $C_{p/q}$ depending on the ratio p/q. Such a conjecture can be proven [2] by using and extending the results described in the previous sections.

The first change with respect to the analysis in subsection 1.3 concerns the multiscale decomposition and the definition of resonance. Only two scales are used in such a case: we say that the propagator associated to a line λ is on scale $n = 0$ if $\nu_\lambda \in q\mathbf{Z} \setminus \{0\}$ and on scale $n = 1$ otherwise (note that no compact support function has to be introduced). Then one defines a cluster as a maximal connected set of lines on scale $n = 1$: a cluster V will be a resonance if $\sum_{v \in V} \mu_v = 0$; note that such a definition allows to a resonance to have an arbitrary number of entering lines. The reason why only two scales are introduced is that only the lines on scale $n = 0$ admit a bad bound: if $n_\lambda = 0$ indeed g_λ behaves as $O(|\nu_\lambda \eta|^2)$ for $\eta \to 0$, while the propagators of the lines on scale $n = 1$ are of order $O(q^2)$, hence they are bounded uniformily in η.

For any resonance V the resonance factor $\mathcal{V}_V(\vartheta)$ will be defined as the right hand side of (20), but now it will be a function of the quantities $\nu_{\ell_1}\eta, \ldots, \nu_{\ell_{m_V}}\eta$, if $\ell_1, \ldots, \ell_{m_V}$ are the lines entering the resonance: as before we shall define the localized resonance factor $\mathcal{L}\mathcal{V}_V(\vartheta)$ as the expansion up to first order of $\mathcal{V}_V(\vartheta)$ and the renormalized resonance factor $\mathcal{R}\mathcal{V}_V(\vartheta)$ as the remaining part.

By neglecting the resonant lines one can easily prove that the number N_0^* of nonresonant lines on scale $n = 0$ is bounded by $N_0^* \leq \lfloor k/q \rfloor$; see subsection 4.3. Then the cancellations mechanism and the bound on the renormalized trees can be discussed essentially as in subsections 1.4 and 1.5 (of course the resummation families have to be enlarged [2] as there can be several lines entering the resonances) and, as in the previous case, one endly shows that there exists a constant $\tilde{C}$ such that

$$\left| u_\nu^{(k)} \right| < \tilde{C}^k \eta^{2k/q}. \tag{39}$$

As $u_\nu^{(k)} = A_\nu \eta^{2k/q} + o(\eta^{2k/q})$, with A_ν a nonvanishing constant, it follows that the limit (37) exists (and it is not trivial). To prove that such a limit solves (38), one simply compare the tree expansions of the limit function (37) and of the solution of (38) and verifies that they coincide.

The above discussion can also apply to the semistandard map: in such a case the bound $N_0^* \leq \lfloor k/q \rfloor$ is sufficient (see comments in subsection 2.1) to draw the same conclusions, *i.e.* the existence of the limit function and the fact that it solves a differential equation differing from (30) inasmuch the sine function is replaced by $\exp\left[iq\left(\alpha+\bar{u}\right)\right]$: this result has been recently proved with other methods [5].

3.5 Homoclinic angles and Arnol'd diffusion

Let us consider the Hamiltonian (34), with $\ell = 2$ and $\underline{\omega} = (\omega_1\sqrt{\eta}, \omega_2/\sqrt{\eta})$, with $\eta \to 0$ and ω_1, ω_2 independent of η. Then the system is a three time scale system, as the characteristic time of the pendulum is of order $O(1)$ in η (if the Lyapunov exponents $\pm g$ are of order $O(1)$), while the angles of the two rotators are slow the first one and fast the second one.

Then one can prove [15] that if $\varepsilon = \eta^c$, with c large enough (and depending on N_0), the homoclinic angles between the stable and unstable manifolds of a 1-codimension invariant torus are not vanishing and can be bounded from below. In fact in such a case the first order of perturbation theory (*Mel'nikov integral*) dominates. The result is nontrivial: in fact the first order has size $O(\eta^{D_1} e^{-D_2/\sqrt{\eta}})$, for some constants D_1, D_2, and, because of the exponential decay factor, it is not obvious that the terms of order strictly larger than one are really smaller than the first one. To verify this one needs a detailed analysis of the perturbative series, exploiting once more the deep analogy with QFT (for instance a Dyson type equation is used along the proof [15]), in order to show that the exponential factor $e^{-D_2/\sqrt{\eta}}$ appears to all orders.

One proves also [17] that the gaps between invariant tori persisting under the perturbation are "smaller" than the homoclic angles, so making possible the existence of a transition chain between the tori. We remark that also in such a case the Lindstedt series analysis allows us to obtain a better bound on the dependence on η of the radius of convergence of the invariant tori: under the hypothesis of a Diophantine condition with constant $C_0 = O(e^{-D_3/\sqrt{\eta}})$, $D_3 > 0$, one finds a bound $|\varepsilon| < \varepsilon_0 = O(\eta^2)$ instead of the bound $|\varepsilon| < \varepsilon_0' = O(\eta^{9/2+})$ (where $+$ denotes logarithmic corrections) which follows from a classical analysis using KAM-type arguments [16].

4 Appendices

4.1 Proof of Siegel-Bryuno's lemma

We give here the proof of the first bound in (18). The proof of the second bound can be derived from the continuous time case [21] following the same lines. [Note that, with respect to the quoted paper, here the scale labels are positive integers.]

We prove by induction on the tree order that, if $\bar{N}_n^*(\vartheta)$ is defined as the number of nonresonant lines in ϑ carring a scale label $\geq n$ and $E_n = 2^{(n-2)/\tau}$, then $\bar{N}_n^*(\vartheta) \le 2kE_n^{-1} - 1$ if $\bar{N}_n(\vartheta) \neq 0$.

Let ϑ be a tree of order k. If ϑ has the root line with scale $< n$ then, calling $\vartheta_1, \vartheta_2, \ldots, \vartheta_m$ the subtrees of ϑ emerging from the highest node of ϑ and with orders $k_j > E_n$ lines, one has $\bar{N}_n^*(\vartheta) = \bar{N}_n^*(\vartheta_1) + \ldots + \bar{N}_n^*(\vartheta_m)$ and the statement is inductively implied from its validity for $k' < k$ and from the fact that $\bar{N}_n^*(\vartheta) = 0$ if $k < E_n$, which follows from the Diophantine condition (9). If the root branch has scale label $\ge n$ one has $\bar{N}_n^*(\vartheta) \le 1 + \bar{N}_n^*(\vartheta_1) + \ldots + \bar{N}_n^*(\vartheta_m)$; if $m = 0$ the statement is trivial and if $m \ge 2$ the statement is again inductively implied by its validity for $k' < k$. If $m = 1$ we once more have a trivial case unless $k_1 > k - E_n/2$. But in the latter case, it turns out that the root line of ϑ_1 is either a resonant line or it has scale $< n$. Accepting the last statement one has $\bar{N}_n^*(\vartheta) = 1 + \bar{N}_n^*(\vartheta_1) = 1 + \bar{N}_n^*(\vartheta'_1) + \ldots + \bar{N}_n^*(\vartheta'_{m'})$, where $\vartheta'_1, \ldots, \vartheta'_{m'}$ are the m' subtrees emerging from the highest node of ϑ'_1 and with orders $k'_j > E_n$. Going once more through the analysis the only nontrivial case is if $m' = 1$ and in that case $\bar{N}_n^*(\vartheta'_1) = \bar{N}_n^*(\vartheta''_1) + \ldots + \bar{N}_n^*(\vartheta''_{m''})$, where $\vartheta''_1, \ldots, \vartheta''_{m'}$ are the m'' subtrees emerging from the highest node of ϑ'_1, and one can iterate further the analysis until one reaches either a trivial case or a tree $\tilde{\vartheta}$ of order $\tilde{k} < k - E_n/2$.

It remains to check that, if $k_1 > k - E_n/2$, then the root line of ϑ_1 has scale $< n$, unless it is entering a resonance. Suppose that the root line of ϑ_1 has scale $\ge n$ and is not entering a resonance. Note that $\|\omega\nu_{\lambda_{v_0}}\| \le C_0 2^{-n}$, $\|\omega\nu_{\lambda_{v_1}}\| \le C_0 2^{-n}$, if v_0 and v_1 are the highest nodes of ϑ and ϑ_1 respectively. Hence $\delta \equiv \|\omega(\nu_{\lambda_{v_0}} - \nu_{\lambda_{v_1}})\| \le C_0 2^{-(n-1)}$ and the Diophantine condition implies that either $|\nu_{\lambda_{v_0}} - \nu_{\lambda_{v_1}}| > 2^{(n-1)/\tau}$ or $\nu_{\lambda_{v_0}} = \nu_{\lambda_{v_1}}$. But $k - k_1 < E_n/2$ implies $\|\omega\nu_\lambda\| \ge C_0 2^\tau 2^{-(n-2)}$ for each line λ preceding the root line and not contained in ϑ_1, so that the set composed by such lines, if $\nu_{\lambda_{v_0}} = \nu_{\lambda_{v_1}}$ and $n_{\lambda_{v_1}} \ge n$, would be a resonance: therefore the latter case has to be discarded as we are not considering resonances. Then $k - k_1 < E_n/2$ would imply $\delta > C_0 2^\tau 2^{-(n-2)} = 2^{\tau+1} C_0 2^{-(n-1)}$, which is contradictory with the above opposite inequality.

Note also that if we used that, for a line λ on scale n, $\|\omega\nu_\lambda\| \le C_0 2^{-(n-1)}$ (as it can be for renormalized trees) the same conclusions would have been drawn, so that the bound $\bar{N}_n^*(\vartheta) \le 2kE_n^{-1} - 1$ holds also after that the renormalization has been performed.

By noting that $N_n^* \le \bar{N}_n^*(\vartheta)$, the first bound in (18) follows.

4.2 Extension of Siegel-Bryuno's lemma for the Bryuno condition

Under the Bryuno condition, choosing $N_1 = 16$ and $N_2 = 8$, if a line λ is on scale n one has $1/16q_{n+1} \le \|\omega\nu_\lambda\| \le 1/8q_n$. Therefore, with the same notations used in the previous subsection, the bound $\rho(\omega) > D_1 e^{-8B(\omega)}$ is proven by showing that (1) one has

$$N_n^* \le \frac{2k}{q_n}, \qquad p_n \le \frac{2k}{q_n} \tag{40}$$

and (2) that the resonances can be dealt with as in subsection 1.4.

The first assertion can be easily proved by proceeding as in subsection 4.1. The only difference is that one has to prove that $\bar{N}_n^*(\vartheta)$ satisfies the bound $\bar{N}_n^*(\vartheta) < 2k/q_n - 1$ (and that $p_n = p_n(\vartheta)$ satisfies the bound $p_n(\vartheta) < 2k/q_n - 1$). All cases can be discussed inductively as in subsection 4.1, except the nontrivial one, *i.e.* except when the root branch has scale $\geq n$ and $m = 1$. In such a case one has once more a trivial case unless $k_1 > k - q_n/2$. Again, if this happens, then either the root line of ϑ_1 is a resonant line either it is on scale $< n$.

Accepting the last statement one can proceed iteratively as before to conclude that at the end the desired bound is obtained. So it remains to check that, if $k - k_1 < q_n/2$, then the root line of ϑ_1, if not resonant, has to be on scale $< n$. Suppose that this is not true. Then the root lines of ϑ and ϑ_1 are both on scale $< n$ and have different momenta (otherwise $k - k_1 < q_n/2$ would imply that the set of lines preceding the root line and not contained in ϑ_1 be a resonance), so that $\|\omega(\nu_{\lambda_{v_0}} - \nu_{\lambda_{v_1}})\| \leq 1/4q_n$, which implies $k - k_1 > q_{n+1}$ by the properties of the convergents $\{q_n\}$: simply recall [25] that $\|\omega\nu\| > 1/2q_{n+1}$ for any integer ν such that $0 < |\nu| < q_{n+1}$.

The proof of the second assertion is trivial; one can define the resonances by imposing $n \geq n_V + 9$, so that for the shifted momenta one has

$$1/32q_{n+1} \leq \|\omega\nu_\lambda\| \leq 1/4q_n, \tag{41}$$

so that the above argument to prove (40) still applies (the only difference is that now $\|\omega(\nu_{\lambda_{v_0}} - \nu_{\lambda_{v_1}})\| \leq 1/2q_n$).

4.3 Standard map near resonances

We can confine ourselves to the cases $q \geq 2$, as the case $q = 1$ is trivial. For $k < q$, $N_0^*(\vartheta)$ is clearly 0, and for $k = q$, $N_0^*(\vartheta) \leq 1$: in these cases the bound is trivially satisfied.

So consider the case $k \geq q > 1$. If the tree ϑ has the root line either on scale $n = 1$ or on scale $n = 0$ and resonant, then the bound $N_0^*(\vartheta) \leq k/q$ follows inductively. On the other hand, if the root line is on scale $n = 0$ and non resonant, then the lines entering the highest node v_0 cannot be all on scale $n = 0$, otherwise $\mu_{v_0} = 0$ (as $q > 1$), which is not allowed. Then at least one line will have scale $n = 1$: let T be the cluster containing it. The cluster T will have m_T entering lines, with $m_T \geq 0$, and – as we just assumed – it is not a resonance: then $\sum_{v \in T} \mu_v \neq 0$, so there must be at least q nodes inside T. The subtrees entering into T will have, respectively, $k_1, \ldots, k_{m_T}$ nodes, with $q + k_1 + \ldots + k_{m_T} \leq k$ so that, again inductively, $N_0^*(\vartheta) = 1 + N_0^*(\vartheta_1) + \ldots + N_0^*(\vartheta_{m_T}) \leq 1 + (k-q)/q \leq k/q$.

As $N_0^*(\vartheta)$ has to be an integer, then $N_0^*(\vartheta) \leq \lfloor k/q \rfloor$.

4.4 A sharp multiscale decomposition

Instead of the C^∞ partition unity introduced in subsection 1.3, one can also consider a sharp partition, in which the functions χ_n appearing in (11) are replaced by characteristic functions. This has the advantage not to have dependence on the compact support functions in the estimate of the radius of convergence for the Linstedt series. It is also possible to show [6] that the characteristic functions are never derived on the boundary of the support, so that no extra problem arises.

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