Orbital symmetries of first order ODEs

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Tutorial paper for the SPT98 conference

0. Introduction.

In most introductory texts and monographs on symmetries of differential equations, the machinery of local Lie group operations and their prolongations is developed in such a way that it can be applied to all classes of differential equations simultaneously. In this paper, on the other hand, we will concentrate almost exclusively on first order ordinary differential equations, which can be approached in a way that uses only basic properties of differential equations and their solutions. The aim is to provide a quick and elementary introduction, intended mostly for novices, and perhaps to serve as a starting point for more in-depth study.

There are several books on symmetries of differential equations worth reading. We only mention the two monographs [11, 12] by Olver, and Gaeta's book [4]. The basic contents of this paper can also be found in the article [15]. The references listed at the end are not intended to, and do not, provide a complete picture on what is going on in this field. As happens all too often, there is an outrageous bias towards publications by the author of this text.

1. Basics.

Consider an autonomous first order ordinary differential equation $\dot{x} = f(x)$ on the open and connected set $U \subseteq \mathbf{R}^n$. (All functions, vector fields etc. are assumed to be infinitely differentiable, unless specified otherwise.) Then it is known that every initial value problem $\dot{x} = f(x)$, $x(0) = y$ has a unique maximal solution, which we will denote $F(t, y)$. (We will refer to F as the general solution, or flow, of this differential equation.) Moreover, there is a maximal open subset $U_f \subseteq \mathbf{R} \times \mathbf{R}^n$, which contains $\{0\} \times U$, such that F is defined and infinitely differentiable on U_f.

Differentiability implies, for instance, that the partial derivative $DF(t, y)$ with respect to y satisfies the linear differential equation

$$\frac{\partial}{\partial t} DF(t, y) = Df(F(t, y))DF(t, y),$$

which is sometimes called the variational equation of $\dot{x} = f(x)$. There are some noteworthy properties of F. By definition, $F(0, y) = y$ for all $y \in U$, and the uniqueness property of the solution implies

$$F(t_1, F(t_2, y)) = F(t_1 + t_2, y)$$

where defined. Moreover, $F(-t, F(t, y))$ exists whenever $F(t, y)$ exists, and then $F(-t, F(t, y)) = y$ follows immediately. Thus, F defines a local one-parameter group of transformations on U_f, and f is said to generate this local one-parameter group. Conversely, it is not hard to show that every local one-parameter transformation group is the general solution of some differential equation $\dot{x} = g(x)$. See Chapter 1 of [11] for more details. (In the complex case, with analytic functions and vector fields, much of what was said above remains true. One serious difference is that there is no longer a well-defined "maximal region of existence" (as a subset of $\mathbf{C}$) for solutions, and a similar problem arises with U_f. But locally this is irrelevant.)

1.1. Solution-preserving maps and symmetries.

Suppose that one also has a differential equation $\dot{x} = h(x)$ on an open and connected subset V of $\mathbf{R}^n$, and a map $\Phi : \tilde{U} \to V$, where $\tilde{U}$ is an open subset of U. Then Φ is called *solution-preserving* from $\dot{x} = f(x)$ to $\dot{x} = h(x)$ if and only if $\Phi(z(t))$ is a solution of the latter equation whenever $z(t)$ is a solution of the former. (If Φ is solution-preserving, then the vector fields f and h are sometimes called *Φ-related.*)

There is a readily verifiable (and easily provable) criterion: The map Φ is solution-preserving from $\dot{x} = f(x)$ to $\dot{x} = h(x)$ if and only if the identity $D\Phi(x)f(x) = h(\Phi(x))$ holds in $\tilde{U}$. (Recall that we assume differentiability of Φ. Otherwise, the matter of solution-preserving maps gets a lot more tricky.)

A natural question arises in this context: Does there exist a (nontrivial) solution-preserving map from $\dot{x} = f(x)$ to a simpler equation (or the other way round)? Locally this is the case near every nonstationary point. More precisely, for every nonstationary point z of f (i.e. $f(z) \neq 0$) there is a neighborhood $\tilde{U}$ and an invertible solution-preserving map $\Psi : \tilde{U} \to U$ from $\dot{x} = f(z)$ to $\dot{x} = f(x)$. (Note that an equation with constant right-hand side is unproblematic.) This is the content of the *straightening theorem*; see, for instance, [11], Prop. 1.29. We also note that it is possible to construct Ψ if the general solution F is given explicitly.

A *symmetry* of $\dot{x} = f(x)$ is defined as a solution-preserving map $\Phi : \tilde{U} \to U$ from $\dot{x} = f(x)$ to itself, with the additional property that $D\Phi(x)$ is invertible for all $x \in \tilde{U}$. (Thus, symmetries are locally invertible. It seems unnecessary to require global invertibility.)

1.2. Orbital symmetries.

Suppose that $\dot{x} = f(x)$ and $\dot{x} = h(x)$ are both given on U. Under what circumstances do these equations have the same solution orbits (possibly with different parameterizations)? Clearly, the set N of stationary points of f must be equal to the set of stationary points of h. Moreover, a necessary condition is the existence of a function $\mu : U \setminus N \to \mathbf{R} \setminus \{0\}$ such that $h(x) = \mu(x) f(x)$ for all x. To see this, note that the tangent vectors to the solutions through any given point of $U \setminus N$ must be linearly dependent, since the curves differ only in their parameterizations. Then existence (and differentiability) of μ follows easily.

Conversely, if $\mu : V \to \mathbf{R} \setminus \{0\}$ is given on some open subset V of U, then the restrictions of $\dot{x} = f(x)$ and $\dot{x} = \mu(x) f(x)$ to V have the same solution orbits, because a solution $z(t)$ of $\dot{x} = f(x)$ can be reparameterized to yield a solution of the second equation: The "ansatz" $w(t) = z(\tau(t))$ yields the condition $\dot{\tau}(t) = \mu(z(\tau(t)))$, and this equation has a unique solution with $\tau(0) = 0$.

Now, an *orbital symmetry* of $\dot{x} = f(x)$ is a solution-preserving map Φ from $\dot{x} = f(x)$ to some equation with the same solution orbits, which (as above) has invertible derivative at every point where it is defined.

1.3. Examples.

• The normalized Euler equation for the asymmetric top with no external forces

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= x_3 x_1 \\ \dot{x}_3 &= -x_1 x_2, \end{aligned}$$

briefly $\dot{x} = f(x)$, admits $\Phi(x) := (x_2, x_1, x_3)^t$ as a symmetry (indeed, $D\Phi(x) f(x) = f(\Phi(x))$ is readily verified), while for $\Psi(x) := (-x_1, x_3, x_2)^t$ one computes $D\Psi(x) f(x) = -f(\Psi(x))$, and thus Ψ is a proper orbital symmetry of this equation.

• Rotationally symmetric equations in the plane: Let γ_1, γ_2 be infinitely differentiable on $[0, \infty[$. Then every rotation

$$x \mapsto \begin{pmatrix} x_1 \cos\alpha + x_2 \sin\alpha \\ -x_1 \sin\alpha + x_2 \cos\alpha \end{pmatrix} \quad (\alpha \in \mathbf{R})$$

is a symmetry of

$$\dot{x} = \gamma_1(x_1^2 + x_2^2) \cdot x + \gamma_2(x_1^2 + x_2^2) \cdot \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

• Quasi-homogeneous systems in the plane: Let $q > 1$ be an integer. Then the system

$$\begin{aligned} \dot{x}_1 &= \alpha x_2 + \beta x_1^q \\ \dot{x}_2 &= \gamma x_1^{q-1} x_2 + \delta x_1^{2q-1} \end{aligned}$$

(with real parameters $\alpha, \ldots, \delta$) admits every scaling transformation

$$x \mapsto \begin{pmatrix} x_1 \lambda \\ x_2 \lambda^q \end{pmatrix} \quad (\lambda > 0)$$

as a proper orbital symmetry.

1.4. The connection to things usually found in books.

Usually, following Lie's original approach, one considers non-autonomous equations

$$\dot{x} = \tilde{f}(t, x) \quad (\tilde{f} \text{ defined on } V \subseteq \mathbf{R} \times \mathbf{R}^n),$$

and a locally invertible map Ψ from some open subset of V to V. (Thus, one has a transformation of "(t, x)-space". Denote $\Psi(t, x) = \left(\begin{smallmatrix} \tilde{t}(t,x) \\ \tilde{x}(t,x) \end{smallmatrix}\right)$.) By definition, Ψ is a symmetry of this differential equation if and only if Ψ transforms every solution $z(t)$ of $\dot{x} = \tilde{f}(t, x)$ to a solution. This involves some subtleties (in particular, one has to require that $\tilde{x}(t, z(t))$ can be re-parameterized by $\tilde{t}(t, z(t))$), but these problems vanish as soon as one chooses the proper setting: $\dot{x} = \tilde{f}(t, x)$ is equivalent to the autonomous equation

$$\begin{aligned} \dot{x}_0 &= 1 \\ \dot{x} &= \tilde{f}(x_0, x) \end{aligned} \tag{1}$$

on V and *Ψ is a symmetry of $\dot{x} = \tilde{f}(t, x)$ in the sense just specified if and only if Ψ is an orbital symmetry of the associated autonomous system.* Indeed, Ψ is solution-preserving from (1) to some equation by local invertibility, and if Ψ is a symmetry of $\dot{x} = \tilde{f}(t, x)$ then the solutions of the new autonomous equation are just the solutions of the old one, up to parameterization. In other words, Ψ is then an orbital symmetry of (1).

If the theory of (local) Lie groups and their operations is developed in a general setting for various applications, as is the case in most books, then there is not much incentive to take an alternative approach to the special class of first order ordinary differential equations, since they fit nicely into the general picture. But the direct approach presented here is easily accessible and very simple.

2. Local one-parameter groups of (orbital) symmetries.

One of Lie's fundamental ideas was to consider local one-parameter groups of symmetries or orbital symmetries. Before going into details, let us introduce a few more notions: Given a vector field f and a real-valued function ϕ on U, the *Lie derivative* of ϕ with respect to f is the function $L_f(\phi)$ defined by $L_f(\phi)(x) := D\phi(x)f(x)$. (In fact, most people nowadays would say that L_f is the vector field, but in this paper we will stick to ancient terminology.) One says that ϕ is a *first integral* of f if $L_f(\phi) = 0$. If g is another vector field then the *Lie bracket* of f and g is defined by $[f, g](x) := Dg(x)f(x) - Df(x)g(x)$. (Note that each L_f is a derivation of the algebra of real-valued functions on U, and it is known that every derivation of this algebra can be written in this form. The bracket $[f, g]$ corresponds to the commutator of L_f and L_g, and therefore the vector fields on U, with this bracket and the usual addition and multiplication by scalars, form a Lie algebra. See Olver [11] and Helgason [7] for more.)

2.1. The basic lemma.

Let f, g be vector fields on U, and denote by G the general solution of $\dot{x} = g(x)$. Then one has the identity

$$\frac{\partial}{\partial s}\left(DG(s,y)^{-1}f(G(s,y))\right) = DG(s,y)^{-1}\left[g,f\right](G(s,y)).$$

(As before, DG denotes the partial derivative with respect to y.)

To prove this, recall $\frac{\partial}{\partial s}DG(s,y) = Dg(G(s,y))DG(s,y)$, which implies

$$\frac{\partial}{\partial s}\left(DG(s,y)^{-1}\right) = -DG(s,y)^{-1}Dg(G(s,y)).$$

(If $Y(t)$ solves a linear matrix equation $\dot{X} = AX$ then $Y(t)^{-1}$ solves $\dot{X} = -XA$.) The desired result now follows with the product rule, by a straightforward computation.

2.2. The basic criterion.

Every element of the local one-parameter group generated by g is a symmetry (resp. an orbital symmetry) of $\dot{x} = f(x)$ if and only if $[g,f] = 0$ (resp. $[g,f] = \lambda f$ with some real valued function λ, possibly except for stationary points of f).

Let us prove the easier part of this: By the criterion for solution-preserving maps, each $G(s,\cdot)$ is a symmetry if and only if $DG(s,y)^{-1}f(G(s,y)) = f(y)$ for all y. Since $G(0,\cdot)$ is the identity map, this is equivalent to $\frac{\partial}{\partial s}\left(DG(s,y)^{-1}f(G(s,y))\right) = 0$, hence $[g,f] = 0$ by the basic lemma.

As for orbital symmetries, use the basic lemma and the characterization that $DG(s,y)^{-1}f(G(s,y))$ and $f(y)$ must be linearly dependent in $\mathbf{R}^n$ for every non-stationary point y.

Sometimes we will call the generator of a local one-parameter group of (orbital) symmetries an infinitesimal (orbital) symmetry.

2.3. The connection to things usually found in books.

Consider a non-autonomous equation $\dot{x} = \tilde{f}(t,x)$ in a subset of $\mathbf{R}\times\mathbf{R}$, and an infinitesimal symmetry $g = \binom{g_0}{g_1}$, thus an infinitesimal orbital symmetry of the associated autonomous equation. The condition, according to (2.2), is

$$\begin{pmatrix} 0 & 0 \\ \frac{\partial \tilde{f}}{\partial x_0} & \frac{\partial \tilde{f}}{\partial x_1} \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} - \begin{pmatrix} \frac{\partial g_0}{\partial x_0} & \frac{\partial g_0}{\partial x_1} \\ \frac{\partial g_1}{\partial x_0} & \frac{\partial g_1}{\partial x_1} \end{pmatrix}\begin{pmatrix} 1 \\ \tilde{f} \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ \tilde{f} \end{pmatrix},$$

equivalently

$$\begin{aligned} -\frac{\partial g_0}{\partial x_0} - \frac{\partial g_0}{\partial x_1}\tilde{f} &= \lambda; \\ \frac{\partial \tilde{f}}{\partial x_0}g_0 + \frac{\partial \tilde{f}}{\partial x_1}g_1 - \frac{\partial g_1}{\partial x_0} - \frac{\partial g_1}{\partial x_1}\tilde{f} &= \lambda\tilde{f}. \end{aligned}$$

Substitute the expression for λ into the second equation to obtain the equivalent condition

$$\frac{\partial \tilde{f}}{\partial x_0} g_0 + \frac{\partial \tilde{f}}{\partial x_1} g_1 - \frac{\partial g_1}{\partial x_0} - \frac{\partial g_1}{\partial x_1} \tilde{f} + \frac{\partial g_0}{\partial x_0} \tilde{f} + \frac{\partial g_0}{\partial x_1} \tilde{f}^2 = 0.$$

Compare this to formula (2.76) on p. 134 in [11]. Up to notation, the conditions are identical (as they should be), but note that we arrived at it by a completely elementary method. (A similar observation holds for first order ODEs in arbitrary dimension.)

The approach to infinitesimal symmetries via infinitesimal orbital symmetries of the associated autonomous equation is not new. It can be found, for instance, in Helgason [7], Ch. II, §8, and Exercise 2.19 on p. 182 in [11] also contains all the necessary ingredients.

The distinction "autonomous vs. non-autonomous" is, of course, to some extent arbitrary. For instance, one may consider an autonomous equation as non-autonomous and thus look at "non-autonomous symmetries" of an autonomous equation. While this may be useful in some instances, it seems unlikely that repeating the procedure could be of much help. (After all, one usually does not voluntarily increase the dimension of a problem!)

2.4. Examples.

• Quasi-homogeneous equations in the plane: Let $q > 1$ be an integer, and $\alpha, \ldots, \delta$ real numbers. For

$$f(x) := \begin{pmatrix} \alpha x_2 + \beta x_1^q \\ \gamma x_1^{q-1} x_2 + \delta x_1^{2q-1} \end{pmatrix} \quad \text{and} \quad g(x) := \begin{pmatrix} x_1 \\ q x_2 \end{pmatrix}$$

one finds $[g, f] = (q-1)f$, thus g generates a local one-parameter group of orbital symmetries for this equation.

Solving the linear differential equation $\dot{x} = g(x)$ yields $G(s, y) = (y_1 e^s, y_2 e^{qs})^t$, and thus we have the orbital symmetries from (1.3).

• The matrix Riccati equation: For $m \times m$-matrices a and x, let $f_a(x) := xax$, and consider the vector fields f_a on $\mathbf{R}^{m,m}$. Using $Df_a(x)y = xay + yax$, one finds that $[f_a, f_b] = 0$ for all matrices a, b, and thus a homogeneous matrix Riccati equation $\dot{x} = xax$ admits plenty of infinitesimal symmetries. One may see this as one of the reasons why matrix Riccati equations behave better than arbitrary quadratic equations.

• The predator-prey equation of Volterra and Lotka: Let $\alpha, \ldots, \delta$ be positive real numbers, and

$$f(x) := \begin{pmatrix} x_1(\alpha - \beta x_2) \\ x_2(-\gamma + \delta x_1) \end{pmatrix}.$$

Then

$$g(x) := \begin{pmatrix} \frac{x_1 x_2}{\alpha\delta x_1 - \beta\gamma x_2} \\ \frac{-\delta x_1 x_2}{\alpha\beta\delta x_1 - \beta^2 \gamma x_2} \end{pmatrix}$$

defines an infinitesimal orbital symmetry of $\dot{x} = f(x)$.

2.5. What is so special about infinitesimal orbital symmetries?

(i) From a practical point of view, verifying the conditions from (2.2) for infinitesimal (orbital) symmetries is definitely easier than checking the conditions for (orbital) symmetries.
(ii) An attempt to determine infinitesimal orbital symmetries of a given vector field stands a better chance of succeeding than an attempt to determine orbital symmetries. (Most people prefer linear equations to nonlinear ones, and there are numerous examples to illustrate that things are really easier on the infinitesimal level. However, it has to be emphasized that the infinitesimal problem is still formidable.)
(iii) Infinitesimal (orbital) symmetries allow a systematic reduction and simplification of the problem. See §3 for more.
(iv) As Lie noted, one can turn the problem around to start with a (harmless) vector field g and determine all the differential equations that admit g as an infinitesimal orbital symmetry. This strategy will produce differential equations whose solution is accessible. See §4 for more. Lie himself showed how this approach explained why many (up to then seemingly unrelated) classes of differential equations in the plane could be integrated with the help of elementary functions.

3. Reduction.

If $\dot{x} = f(x)$ admits an infinitesimal orbital symmetry g whose general solution can be determined explicitly then it is possible to reduce the dimension of the problem, up to finding antiderivatives.

3.1. Lie's reduction theorem.

(a) Let Ψ be locally invertible and solution-preserving from $\dot{x} = (1, 0, \ldots, 0)^t$ to $\dot{x} = g(x)$. Then Ψ is also solution-preserving from the equation $\dot{x} = f^*(x)$ to $\dot{x} = f(x)$, where $f^*(x) = D\Psi(x)^{-1} f(\Psi(x))$.
(b) If $[g, f] = 0$ then f^* does not depend on x_1. Thus, solving $\dot{x} = f^*(x)$ amounts to solving an equation in dimension $n-1$ and an integration problem.
(c) If $[g, f] = \lambda f$ then there is a real-valued function μ and a vector field f^{**} that is independent of x_1 such that $f^* = \mu f^{**}$. Thus, solving $\dot{x} = f^*(x)$ amounts to solving an equation in dimension $n-1$ and two integration problems. (Note that one finds the solution of f^{**} from the solution of f^* by finding antiderivatives; see (1.2).)

There is nothing to say about (a). The key to the proofs of (b) and (c) is the fact that solution-preserving maps respect Lie brackets. Thus $[g, f] = 0$ implies $[(1, 0, \ldots, 0)^t, f^*] = 0$, and this is simply $\partial f^*/\partial x_1 = 0$.

Likewise, $[g, f] = \lambda f$ implies $\partial f^*/\partial x_1 = \lambda^* f^*$. Now, if μ satisfies $\partial \mu/\partial x_1 = \lambda^* \mu$ then $(\partial/\partial x_1)(\mu^{-1} f^*) = 0$.

For practical purposes it is noteworthy that in the situation of (c) the vector fields f^* and f^{**} have the same solution orbits, and that one may therefore find the solution orbits of $\dot{x} = f(x)$ from those of f^{**}.

Let us look at one example: The quasi-homogeneous equations from (2.4) admit the infinitesimal orbital symmetry $g(x) = (x_1, qx_2)^t$. The map $\Psi(x) = \left(e^{x_1-1}, x_2 e^{q(x_1-1)}\right)^t$ satisfies $D\Psi(x)(1,0)^t = g(\Psi(x))$. Thus, by the recipe of the reduction theorem, let

$$f^*(x) := D\Psi(x)^{-1} f(\Psi(x)) = e^{(q-1)(x_1-1)} \begin{pmatrix} \alpha x_2 + \beta \\ -q\alpha x_2^2 + (\gamma - q\beta)x_2 + \delta \end{pmatrix}.$$

Obviously finding the solution of

$$\begin{aligned} \dot{x}_1 &= \alpha x_2 + \beta \\ \dot{x}_2 &= -q\alpha x_2^2 + (\gamma - q\beta)x_2 + \delta \end{aligned}$$

only involves (elementary) integration problems, and this is already sufficient to determine the solution orbits of $\dot{x} = f(x)$.

3.2. Infinitesimal orbital symmetries and integrating factors.

In dimension two, integrating factors and nontrivial infinitesimal orbital symmetries are just two incarnations of the same property. Let us first note that obviously f itself is an infinitesimal symmetry of f, and that every (nonzero) scalar-valued function σ yields the infinitesimal orbital symmetry σf of $\dot{x} = f(x)$. This simply reflects the well-known fact that any translation of time is a solution-preserving map of an autonomous equation. We will call these infinitesimal symmetries and orbital symmetries trivial. The relation to integrating factors can be expressed as follows (for a proof see [11], Theorem 2.48):

Let g be a nontrivial infinitesimal orbital symmetry of $\dot{x} = f(x)$. Then $\rho(x) := \det\left(g(x), f(x)\right)^{-1}$ is an integrating factor of f, i.e. $\mathrm{div}(\rho f) = 0$.

Note that locally $\rho f = \left(-\frac{\partial \mu}{\partial x_2}, \frac{\partial \mu}{\partial x_1}\right)^t$ for some function μ, and that μ is therefore a first integral of $\dot{x} = \rho(x) f(x)$, hence also of $\dot{x} = f(x)$. Thus knowledge of an integrating factor reduces the problem of finding the solution orbits (more precisely, the level sets of μ) to a problem of integration theory.

Conversely, for an integrating factor σ let g satisfy $\det\left(g(x), f(x)\right) = \sigma(x)^{-1}$. It is not hard to verify that g is then an infinitesimal orbital symmetry.

For example, the Volterra-Lotka equation in (2.4) admits the integrating factor $\rho(x) = (x_1 x_2)^{-1}$.

3.3. Reduction in the "multi-parameter" case.

Suppose that $g_1, \ldots, g_r$ (with $r \geq 2$) are vector fields satisfying $[g_i, f] = \lambda_i f$ for all i. Naturally one expects that this should reduce the problem even more. Unfortunately, things are not that straightforward.

Let us first clarify the algebraic background. It is a direct consequence of the Jacobi identity that whenever g and h are infinitesimal orbital symmetries (or infinitesimal symmetries) then this is also the case for their Lie bracket $[g, h]$. Therefore the infinitesimal (orbital) symmetries of $\dot{x} = f(x)$ form a Lie algebra, and if $g_1, \ldots, g_r$ are known then it is natural to consider the subalgebra they generate. This subalgebra will not, in general, be finite dimensional, and it turns out that it is more appropriate to consider (locally) the involution system generated by the g_i: It is said that $g_1, \ldots, g_r$ are in involution on the open set $\tilde{U} \subseteq U$ if there are (differentiable) functions μ_{ijk} such that

$$[g_i, g_j] = \sum \mu_{ijk} g_k$$

for all i and j. It is possible to extend any finite set of vector fields to an involution system by forming Lie brackets, and thus we may assume that the infinitesimal orbital symmetries $g_1, \ldots, g_r$ are in involution, and furthermore that $g_1(y), \ldots, g_r(y)$ span an r-dimensional subspace for every $y \in \tilde{U}$. (This can be achieved by reducing the number of vector fields, if necessary, possibly at the expense of replacing $\tilde{U}$ by a smaller set.) Then, by a theorem of Frobenius (see [11], Theorem 1.40), there are locally $n - r$ functionally independent common first integrals $\gamma_1, \ldots, \gamma_{n-r}$ of the g_i, and furthermore every common first integral of the g_i is a function of the γ_j.

The following reduction theorem can be found, for instance, in Hermann[8]. Basically, like so much in this field, it goes back to Lie.

Suppose that $g_1, \ldots, g_r$ are in involution on $\tilde{U}$, with $r < n$, and that $\gamma_1, \ldots, \gamma_{n-r}$ are functionally independent common first integrals of the g_i. If all $[g_i, f] = 0$ then the map

$$\Gamma : \tilde{U} \to \mathbf{R}^{n-r}, \quad x \mapsto \begin{pmatrix} \gamma_1(x) \\ \vdots \\ \gamma_{n-r}(x) \end{pmatrix}$$

is solution-preserving from $\dot{x} = f(x)$ to some differential equation in a subset of $\mathbf{R}^{n-r}$.

The proof rests on the fact that each $L_f(\gamma_j)$ is again a common first integral of the g_i, which follows directly from

$$L_{[g_i, f]}(\gamma_j) = L_{g_i} L_f(\gamma_j) - L_f L_{g_i}(\gamma_j) = L_{g_i} L_f(\gamma_j).$$

Thus, each $D\gamma_j(x) f(x)$ can be expressed as a function of $\gamma_1, \ldots, \gamma_{n-r}$, and therefore
$D\Gamma(x) f(x) = h(\Gamma(x))$ for a suitable vector field h. With the criterion for solution-preserving maps, we are finished.

The case of infinitesimal orbital symmetries is, in a sense, reducible to the case of infinitesimal symmetries. In fact, if $[g_i, f] = \lambda_i f$ for all i, and if there is a common first integral of the g_i which is not a first integral of f, then there is (locally, near nonstationary points of f) a function $\sigma \neq 0$ such that all $[g_i, \sigma f] = 0$ (and we know that f and σf have the same orbits); see [16], Lemma 2.3. This is not really satisfactory, since the interesting local behavior near stationary points is not accessible in this way.

In general, one should not expect to obtain a "reduction of dimension modulo integration problems", similar to the situation for one-parameter groups, in the multi-parameter case. Such a property is related to certain ("solvability") conditions for the involution system.

The structure of the set of infinitesimal orbital symmetries of a vector field is discussed in [4] (e.g. Ch. I, §12). The paper [16] includes a survey of classical results (mostly due to Lie and Bianchi) on these matters. One should remark here that the value of the reduction theorems stated above is somewhat limited, since the reduction procedure outlined makes use of non-constructive results (like the inverse function theorem). In this sense, one could say that these reduction theorems are blueprints for the actual work, much of which remains yet to be done.

3.4. Linear orbital symmetries.

For the reduction of differential equations admitting certain linear symmetry groups (for instance compact ones) one has a more sophisticated apparatus to work with. Compact symmetry groups are the focus of attention in the monograph [6] by Golubitsky, Stewart and Schaeffer, and the reduction is also discussed, for example, in Abud and Sartori [1].

The basic ingredients for the reduction are the polynomial invariants of a compact linear group G, i.e. those polynomials ρ which satisfy $\rho \circ T = \rho$ for all $T \in G$. It is known that the algebra of polynomial G-invariants is finitely generated. Let $\phi_1, \ldots, \phi_r$ be a system of generators. Then the following holds:

If every element of G is a symmetry of $\dot{x} = f(x)$ then $\Phi := (\phi_1, \ldots, \phi_r)^t$ is solution-preserving from $\dot{x} = f(x)$ to some equation $\dot{x} = h(x)$ in $\mathbf{R}^r$.

The proof of this is easy (and follows the pattern established in (3.3)) when f is a polynomial: Each polynomial $L_f(\phi_i)$ is again G-invariant, hence can be written as a polynomial in $\phi_1, \ldots, \phi_r$. As for infinitely differentiable vector fields, one needs some serious theoretical machinery, due to Schwarz and Poenaru, see [6]. (The case of differentiable, but not infinitely differentiable, functions and vector fields was recently resolved by Rumberger and Scheurle [14], and in Rumberger's dissertation [13].)

Let us look at a simple example. The Euler equation (see (1.3)) admits a two-element linear symmetry group generated by $x \mapsto (x_2, x_1, x_3)$. The invariant algebra of this group is generated by $\phi_1(x) = x_1 + x_2$, $\phi_2(x) = x_1 x_2$, and $\phi_3(x) = x_3$. Now one has, for instance, $L_f(\phi_1)(x) = x_2 x_3 + x_3 x_1 = \phi_1(x)\phi_3(x)$, and similar computations show that Φ is solution-preserving from $\dot{x} = f(x)$ to the equation

$$
\begin{aligned}
\dot{x}_1 &= x_1 x_3 \\
\dot{x}_2 &= (x_1^2 - 2x_2) x_3 \\
\dot{x}_3 &= -x_2.
\end{aligned}
$$

There is no easily discernible simplification or reduction in this example, and indeed this should not be generally expected in the case of a finite (thus zero-dimensional) group. Still, one may say that ϕ reduces f by taking the symmetries into account. For compact linear symmetry groups of positive dimension one gets a reduction of dimension for the problem. There is an additional subtlety here whenever the group does not admit an algebraically independent set of generators for the invariant algebra. In this case, the reduced vector field lives on an algebraic variety, the so-called discriminant variety of the group; see [6].

Generally, the reduction procedure via invariants of G leads to a certain loss of information about the differential equation (actually, the same can already be said about the procedure from (3.3)). The problem of how to introduce new coordinates that reflect the simplification by symmetries while keeping all the information is addressed by Gaeta in [5].

Concerning a (compact) linear group of orbital symmetries, there is also a natural reduction procedure, as was shown in [9]. Instead of quoting the theorem given there, let us look at one example: The equation

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 - x_3) \\ \dot{x}_2 &= x_2(x_3 - x_1) \\ \dot{x}_3 &= x_3(x_1 - x_2), \end{aligned}$$

briefly $\dot{x} = f(x)$, admits the group S_3 of (linear maps defined by) the permutations of the standard basis as an orbital symmetry group. (The reflections in this group are proper orbital symmetries, satisfying $T^{-1}fT = -f$.) In this case the map

$$\Psi : \mathbf{R}^3 \to \mathbf{R}^4,\ x \mapsto \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1x_2 + x_2x_3 + x_3x_1 \\ x_1x_2x_3 \\ (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \end{pmatrix}$$

is solution-preserving from the given equation to

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= 0 \\ \dot{x}_4 &= -6x_2^2 + x_1^2x_2 + 9x_1x_3, \end{aligned}$$

as is easily verified. Note that this equation can be solved using elliptic functions. The reason why this map (involving invariants and relative invariants of the group) is canonically solution-preserving to some other equation is explained in [9]. One has to emphasize that generally there will not be such a dramatic simplification as in the example presented here, but it indicates the occasional usefulness of orbital symmetries even in the case of finite groups.

4. Finding infinitesimal orbital symmetries.

In the literature there is a remarkable distinction between first order ordinary differential equations and practically everything else, when it comes to computing infinitesimal orbital symmetries. The reason is that for everything else there is at least a foothold to start computations, while in the case of first order ODEs there is no general strategy available.

4.1. The problem.

Let f be given. Finding a vector field g such that $[g, f] = 0$ (or $[g, f] = \lambda f$) amounts to finding a (nontrivial) solution of a system of coupled linear first-order partial differential equations. The problem is not that there are too few such solutions, but that locally, near any nonstationary point, *there are too many.* (This is a simple consequence of the straightening theorem.) For most other classes of differential equations, the symmetry conditions provide sufficiently many restrictions to make the determination of symmetries computationally accessible, while this feature is absent in the case of first order ODEs.

What can be done about this? For instance, one may a priori restrict the class of functions in which the search for symmetries is carried out. (For example, one may look for polynomial infinitesimal orbital symmetries, of some fixed degree, of a given polynomial vector field.) An other approach is to investigate (orbital) symmetries near stationary points, where existence is not guaranteed. This question is related to properties of the Poincaré-Dulac normal form at such a stationary point.

It seems unlikely that the future will bring a universally (and successfully) applicable strategy for first order ODEs, but searching for symmetries subject to certain (appropriately chosen) restrictions will probably yield more, and interesting, results.

4.2. Turning things around.

As Lie noted, starting with the infinitesimal orbital symmetry, and then determining the symmetric vector fields, will bring some success. We will focus our attention to two-dimensional equations, for sake of simplicity. Thus, let g be a vector field for which a straightening function Φ, with $D\Phi(x)g(x) = \binom{1}{0}$, is known. (Essentially, this means that the general solution of $\dot{x} = g(x)$ is known.) Now let $f^*(x) = \mu(x) f^{**}(x_2)$ be a vector field that depends only on x_2, up to multiplication by some scalar-valued function. (As we noted before, these are precisely the vector fields that satisfy $[\binom{1}{0}, f^*] = \lambda^* f^*$ for some λ^*.) Defining $f(x) := D\Phi(x)^{-1} f^*(\phi(x))$, one obtains a vector field satisfying $[g, f] = \lambda f$. Moreover, the reduction theorem (3.1) can be readily applied.

An example may be useful. Let

$$g(x) = \begin{pmatrix} x_1 \\ q x_2 \end{pmatrix} \text{ and } \Phi(x) = \begin{pmatrix} 1 + \ln x_1 \\ x_2 / x_1^q \end{pmatrix}.$$

Then $D\Phi(x)g(x) = \binom{1}{0}$ is easily verified. (Incidentally, Φ is the inverse of the map Ψ from the example in (3.1).) A straightforward computation shows that the vector fields of the form

$$f(x) = \mu(1 + \ln x_1, x_2/x_1^q) \begin{pmatrix} x_1\rho_1(x_2/x_1^q) \\ qx_2\rho_1(x_2/x_1^q) + x_1^q\rho_2(x_2/x_1^q) \end{pmatrix},$$

and only these, admit the infinitesimal orbital symmetry g. Thus we have rediscovered a class of differential equations that is integrable by elementary functions.

4.3. Higher order ODEs.

As shown by a glance into any book on the subject, there is a clear-cut road for computing infinitesimal symmetries of higher order ordinary differential equations, and it also turns out that these form a finite dimensional Lie algebra. On the other hand, it is known from introductory courses on differential equations that higher order equations are equivalent to first order equations in a higher-dimensional space, and one may therefore wonder where the different behavior concerning symmetries comes from. The answer is that the class of admissible symmetries for higher-order equations is usually restricted a priori by geometrical considerations. (The symmetries are required to be induced from "point transformations".) Here we will only look at second order differential equations in one dependent variable. This class of examples contains all the characteristic features of the general case.

Thus, consider the non-autonomous second order equation

$$\ddot{x} = \rho(t, x, \dot{x}),$$

and the equivalent autonomous system

$$\begin{aligned} \dot{x}_0 &= 1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \rho(x_0, x_1, x_2). \end{aligned}$$

We now restrict the search for orbital infinitesimal symmetries to those of the special type

$$g(x_0, x_1, x_2) = \begin{pmatrix} \gamma_0(x_0, x_1) \\ \gamma_1(x_0, x_1) \\ \gamma_2(x_0, x_1, x_2) \end{pmatrix}.$$

(Thus, the (t, x)-space, or "point space", is transformed to itself. In the approach presented here, the issue of prolongations of vector fields does not come into play explicitly. This is one of the cases where one can proceed without needing to really understand the underlying mechanism. Incidentally, for a higher order equation one would continue with the next entries of g in a kind of "echelon form", thus $\gamma_3 = \gamma_3(x_0, \ldots, x_3)$, and so on.) The evaluation of $[g, f] = \lambda f$

yields

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{\partial\rho}{\partial x_0} & \frac{\partial\rho}{\partial x_1} & \frac{\partial\rho}{\partial x_2} \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} - \begin{pmatrix} \frac{\partial\gamma_0}{\partial x_0} & \frac{\partial\gamma_0}{\partial x_1} & 0 \\ \frac{\partial\gamma_1}{\partial x_0} & \frac{\partial\gamma_1}{\partial x_1} & 0 \\ \frac{\partial\gamma_2}{\partial x_0} & \frac{\partial\gamma_2}{\partial x_1} & \frac{\partial\gamma_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \\ \rho \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ x_2 \\ \rho \end{pmatrix}.$$

From the first equation one finds

$$\lambda = -\frac{\partial\gamma_0}{\partial x_0} - \frac{\partial\gamma_0}{\partial x_1} x_2.$$

The second equation then determines γ_2 in terms of γ_0, γ_1 and their partial derivatives. (This is actually a version of the prolongation formula.) The third equation yields, after the dust has settled, a (possibly infinite) system of linear partial differential equations for γ_0 and γ_1, which depends on ρ. (For instance, in the analytic case, setting $\rho = \sum \rho_i(x_0, x_1)x_2^i$, evaluating the third equation and rearranging terms yields a relation $\sum \alpha_i(x_0, x_1)x_2^i = 0$, whence all $\alpha_i = 0$. These relations produce the partial differential equations just mentioned.)

For the general approach (which yields the same conditions), and many examples, see [11], Ch. 2, in particular §2.3 through §2.5 and some of the exercises.

4.4. An example.

Let μ and ν be real numbers. Letting the machinery of (4.3) work on the equation

$$\ddot{x} = \mu\, x\dot{x} + \nu\, x^3,$$

one finds only the infinitesimal symmetry $g = (1, 0, 0)^t$, up to scalar multiples, except for a few particular values of μ and ν. The existence of g reflects the fact that the given equation is autonomous, but does not lead to any reduction beyond this. Here we have the problem with the "point symmetries" of (4.3): Usually, there will be no nontrivial ones.

Things are even worse in this special example, because the equivalent first order equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu\, x_1 x_2 + \nu\, x_1^3 \end{aligned}$$

is quasihomogeneous (see (1.3), with $q = 2$), and therefore can be reduced to integration problems.

Thus, the a priori restrictions of (4.3) allow computations with a definitive result, and of course "point symmetries" are distinguished from a geometrical point of view. But, as this example shows, by focussing only on them one risks overlooking other classes of symmetries that are just as useful. One will probably always be forced into some kind of trade-off between computational accessibility and efficiency in detecting symmetries, but one should certainly try to keep the list of candidates as big as possible.

5. Are orbital symmetries really interesting?

Since the late 19^{th} century, the problem of finding explicit solutions of a given equation $\dot{x} = f(x)$ in terms of elementary functions (with varying definitions of "elementary") has lost some importance, while the study of qualitative properties has gained much attention, and seen enormous progress. Thus, one would be inclined to give little thought to (orbital) symmetries if they were only useful in the context of finding explicit elementary solutions.

However, it is being increasingly recognized that the existence of nontrivial symmetries (or orbital symmetries) has a strong influence on qualitative properties of differential equations.

For instance, it may be said that the whole monograph [6] by Golubitsky, Stewart and Schaeffer is devoted to an investigation of the influence of a compact linear symmetry group on qualitative behavior, in particular in bifurcation problems. They also emphasize the importance of taking these symmetries into account, in order to obtain a correct description of the situation. Moreover, it has long been recognized that symmetries force certain invariant sets upon differential equations. (The keyword "stratification" should be mentioned here. See [4], and [6].) In [16] one can find a general result on invariant sets in the presence of orbital symmetries.

In the local analytic theory of (nondegenerate) stationary points, it has turned out that the existence of nontrivial symmetries is intimately connected to the existence of an analytic transformation into Poincaré-Dulac normal form. (There is always a formal power series transformation, but convergence problems are frequent.) See Markashov [10] and Bruno and Walcher [2] for two-dimensional systems, and generalizations by Cicogna [3] to higher dimensions.

In order to see how proper orbital symmetries may come into play in local theory, consider (for an integer $q > 1$)

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^q \\ \dot{x}_2 &= q x_1^{q-1} x_2 - x_1^{2q-1}. \end{aligned}$$

Using the infinitesimal orbital symmetry, and the corresponding integrating factor, of this quasi-homogeneous equation, one finds, after some computing, an analytic first integral in a neighborhood of the stationary point 0, and from this first integral one can deduce that 0 is a center of this equation. For two-dimensional systems there is a well-developed theory and a lot of tools for discussing qualitative behavior. Using these tools for the given equation will readily show that 0 must be a center or a focus, but one needs additional information to decide whether it is a center. Here the orbital symmetries provide this information.

As these examples indicate, symmetries and orbital symmetries may cause, and at the same time explain, unusual qualitative behavior. It is, above all, for this reason that they need to be studied further.

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