SPT2001
Symmetry and Perturbation Theory

Cala Gonone (NU - Italy)
6–13 May 2001
Abstracts of communications

April 28, 2001

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Simonetta Abenda

Geometry and dynamics of algebraically integrable systems with deficiency: the case of hyperelliptically separable systems

In this talk I consider integrable Hamiltonian systems whose first integrals are rational functions and whose complexified invariant manifolds may be completed to strata (i.e. nonlinear subvarieties) of convenient (generalized) Jacobi varieties associated to algebraic curves.

There are many physically interesting examples under the more restrictive hypothesis that the algebraic curve is hyperelliptic: II Henon-Heiles integrable cases when the potential has degree ≥ 4, the geodesic flow on the ellipsoid in its natural coordinates, certain finite dimensional reductions of the Harry-Dym and of the shallow water equation, integrable generalizations of the Neumann system, etc.

In this talk, I present an overview of the algebro-geometrical and dynamical characterization of such integrable systems. In particular, I consider analogs of the Jacobi-Mumford systems (coordinates and vector fields) on the natural fibering over the base formed by genus g hyperelliptic curves Γ whose fibers are n–dimensional strata W_n of $\mathrm{Jac}(\Gamma)$ $(n \leq g)$. At any point D of W_n, n independent invariant vector fields can be constructed which span the tangent space of W_n at D. The vector fields I construct on W_n are restrictions of the usual Jacobi-Mumford ones on $\mathrm{Jac}(\Gamma)$.

It turns out that h.s.s. may be viewed as constrained a.c.i. integrable Hamiltonians according to the Dirac formalism.I apply the above construction to two families of examples: the generalized Henon-Heiles and the Neumann hierarchies.

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Faridon Amdjadi

O(2) Symmetry, Hopf Bifurcation and Kuramoto-Sivashinsky Equation

In problems with $O(2)$ symmetry, the jacobian of the system on a branch of non-trivial steady state is always singular. While a couple of purely imaginary eigenvalues cross the imaginary axis, the Hopf bifurcation is not of standard type. Canonical coordinates transformation employed by Amdjadi and Gomatam [Phys. Lett. A 231 (1997) 379-388] to deal with such problems. The method cannot be applied for infinite dimensional partial differential equations.

The talk concerns addition of a phase condition to the equations and reducing such problems to the standard form. Application of the method on Kuramoto-Sivashinsky equation will be considered and it will be shown that multiple Hopf bifurcation leading to oscillating waves can occur on a branch with dihedral group of symmetry. It will be shown that these bifurcations are associated with the irreducible representations of group D_n.

Dario Bambusi, S.Graffi, T. Paul

Normal forms methods in quantum mechanics

We present two results obtained by normal forms methods in quantum mechanics.

(1) Consider the Schrödinger equation in one space dimension with a confining potential. Add a small (in a suitable sense) perturbation which depend quasiperiodically on time. We prove by KAM methods that there exists a unitary quasiperiodic transformation that transforms the system into an autonomuos one. As a consequence the perturbed dynamics essentially coincides with the unperturbed one. This has to be confronted with the corresponding classical system where small perturbations induce chaos.

(2) Consider a classical Hamiltonian system $H = p^2/2 + V(q)$ with $V(q)$ having an absolute minimimum in zero. Denote by $\hat{H}$ the corresponding quantized Hamiltonian. When the energy is small the classical Hamiltonian is approximated by harmonic oscillators, and therefore one expects the low lying eigenvalues of $\hat{H}$ to be given by $(n + 1/2) \cdot \omega\hbar$. A better approximation of the classical Hamiltonian is given by its Birkhoff normal form $Z_c(I_i)$ which, in the nonresonant case, is a function of the actions I_i. Is there any relation between the numbers obtained by quantizing the actions, i.e. $Z_c((n_i + 1/2)\omega\hbar)$ and the eigenvalues of $\hat{H}$? We prove that there exists a smooth function $Z_q(I_i, \hbar)$ such that the eigenvalues of $\hat{H}$ in $[0, \hbar^\delta]$ $(\delta > 0)$ are given, up to any order in $\hbar$ by $Z_q((n_i + 1/2)\omega\hbar, \hbar)$, and moreover $Z_q(I_i, 0) = Z_c(I_i)$.

Genrich Belitskii and Alexander Kopanskii

Sternberg-Chen Theorem for Equivariant Hamiltonian Vector Fields

Smooth hamiltonian vector fields with linear symmetries and anti-symmetries are considered. We prove that provided the symmetry group is compact then a smooth conjugacy in Sternberg-Chen Theorem can be chosen canonical and symmetric.

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Massimiliano Berti

A functional analysis approach to Arnold Diffusion

We discuss in the context of nearly integrable Hamiltonian systems a self-contained functional analysis approach to the "splitting of separatrices" and to the "shadowing problem" developed in the papers of M. Berti, P. Bolle: "Diffusion time and splitting of separatrices for nearly integrable isochronous Hamiltonian systems", *Rend. Mat. Acc. Naz. Lincei* **11**, (2000), pp. 235-243; "A functional analysis approach to Arnold diffusion", to appear in *Annales de l'I.H.P, analyse non lineaire*; "Fast Arnold's diffusion in three time scales systems", preprint SISSA 2001.

We apply our method to the problem of Arnold Diffusion for nearly integrable partially isochronous Hamiltonian systems improving known results.

More precisely we consider n-harmonic oscillators weakly coupled with a pendulum through purely spatial perturbations described by the Hamiltonian

$$\mathcal{H}_\mu = \omega \cdot I + \frac{p^2}{2} + (\cos q - 1) + \mu f(\varphi, q),$$

where $(\varphi, q) \in \mathbf{T}^n \times \mathbf{T}^1 := (\mathbf{R}^n / 2\pi \mathbf{Z}^n) \times (\mathbf{R}/2\pi\mathbf{Z})$ are the angle variables, $(I, p) \in \mathbf{R}^n \times \mathbf{R}^1$ are the action variables and $\mu \geq 0$ is a small real parameter.

When $\mu = 0$ the energy $\omega_i I_i$ of each oscillator is a constant of the motion. The problem of *Arnold diffusion* in this context is whether, for $\mu \neq 0$, there exist motions whose net effect is to transfer $O(1)$-energy from one oscillator to the others. In order to exclude trivial drifts of the actions due to resonance phenomena, we assume a standard diophantine condition for the frequency vector ω.

We prove for Hamiltonian $\mathcal{H}_\mu$ general shadowing theorems, which improve known estimates on the diffusion time. Other results like symbolic dynamics can also be obtained by our approach. Secondly we discuss a new method for the splitting of the separatrices, providing general estimates on the Fourier coefficients of some "splitting function".

As applications we consider the following two cases: (a) the frequencies of the harmonic oscillators form a diophantine vector ω of order 1 ("a priori-unstable case"); (b) the frequencies of the harmonic oscillators form a diophantine vector $\omega_\varepsilon = (1/\sqrt{\varepsilon}, \beta\varepsilon^a)$, $a > 0$ and ε is a positive small parameter ("three-time-scales problem"). For three time scales systems, with $n \geq 3$, exploiting the anisotropy of the splitting, we also prove that, along special directions, Arnold diffusion takes place with fast (polynomial) speed, even though the splitting determinant is exponentially small.

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M.S. Bruzón, M.L. Gandarias and J. Ramírez

On symmetries for a Boussinesq equation with nonlinear dispersion

We apply the Lie-group formalism and the nonclassical method due to Bluman and Cole to deduce symmetries of the generalized Boussinesq equation,

$$u_{tt} = au_{xx} + \left(u^{m+1}\right)_{xx} + b\left[u\left(u^m\right)_{xx}\right]_{xx}, \tag{$*$}$$

where a and b are arbitrary constants. For $a = 0$ and $m = 1, 2$, equation (*) describes the vibrations of a purely anharmonic lattice and support travelling structures with a compact support.

We make a full analysis of the symmetry reductions and we prove that the nonclassical method applied to equation (*) leads to new reductions, which cannot be obtained by Lie classical symmetries. Some new exact solutions can be derived. We also have write (*) in a conserved form and new classes of non local symmetries have been obtained.

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Monica Conti

Infinitely many solutions to fourth order superlinear periodic problems

We present a variational approach to the search of multiple T–periodic solutions to a class of fourth order equations

$$u^{iv}(t) - cu''(t) = f(t, u(t)) \qquad t \in (0, T)$$

where $f(t, u)$ is continuous, T–periodic in t and satisfy a superlinear assumption when $|u| \to \infty$. We prove the existence of a T–periodic solution with any prescribed number of zeroes. [Joint work with: Susanna Terracini, Gianmaria Verzini]

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Ana Paula Da Silva Dias

Heteroclinic cycles and wreath product symmetries

We consider the existence and stability of heteroclinic cycles arising by local bifurcation in dynamical systems with wreath product symmetry $Z_2 \wr G$, where Z_2 acts by $+-1$ on R and G is a transitive permutation group. We relate the possibility of such cycles to the existence of non-gradient equivariant vector fields of cubic order.

Using Hilbert series and the software package MAGMA we show that apart from the cyclic groups G (already studied by other authors) only five groups G of degree less than or equal to 7 are candidates for the existence of heteroclinic cycles. We establish the existence of certain types of heteroclinic cycle in these cases by making use of the concept of a subcycle. We also discuss a generalisation of heteroclinic cycles which we call a heteroclinic web. We apply our methods to three examples. [Joint work with B.Dionne and I.Stewart]

Gianne Derks

The symplectic Evans matrix and solitary wave instability

Hamiltonian evolution equations which are equivariant with respect to the action of a Lie group are models for physical phenomena such as oceanographic flows, optical fibres and atmospheric flows, and such systems often have a wide variety of solitary wave or front solutions. In this talk, I will present a symplectic framework for analysing the spectral problem associated with the linearisation about such solitary waves and fronts. At the heart of the analysis is a multi-symplectic formulation of the Hamiltonian wave equation where a distinct symplectic structure is assigned for the time and space directions. This leads to a geometric decomposition and symplectification of the Evans function formulation for the linearisation about solitary waves and fronts. and the introduction of the concept of the symplectic Evans matrix. It can be shown that the zeros of the Evans function correspond to zeros of the determinant of the symplectic Evans matrix.

Restricting the spectral parameter to the real axis, rigorous results can be obtained on the derivatives of the Evans function near the origin, based solely on the abstract geometry of the equations. By combining this with the behaviour of the Evans function for large values of the spectral parameter, a sufficient condition for instability of solitary waves and fronts is obtained.

The theory applies to a large class of solitary waves and fronts including waves which are bi-asymptotic to a nonconstant manifold of states as the spatial variable tends to infinity. To illustrate the theory, it will be applied to a Boussinesq model from oceanography. [Joint work with T. Bridges]

M.L. Gandarias, M.S. Bruzón and J.Ramírez

Symmetry analysis and reductions of the Schwarz-Korteweg-de Vries equation in (2+1) dimensions.

In this paper we make a full analysis of the symmetry reductions of the $(2+1)$-dimensional integrable Schwarz-Korteweg-de Vries equation by using the classical Lie method of infinitesimals. The reduction to systems of partial differential equations in (1+1) are obtained from the optimal system of subalgebras. These systems admit symmetries which lead to further reductions, i.e. to systems of ordinary differential equations. Further, we present a brief analysis for some types of particular solutions.

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F. Güngör

Symmetries and invariant solutions of the two-dimensional Burgers equation with variable coefficient.

We study symmetries of a 2+1-dimensional Burgers equation with variable coefficient. We show that the equation admits an infinite-dimensional Lie algebra as the algebra of its symmetry group which does not have a Virasoro structure whose presence characterize integrability for PDEs in more than 1+1-dimensions. We give a classification of its low-dimensional subalgebras and obtain reduced ODEs. In contrast to an integrable PDE, its reductions to ODEs do not lead to Painlevé type equations. We pick out of them those equations which pass the Painlevé test and obtain their exact solutions.

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Hans-Peter Kruse

A two-dimensional version of the Camassa-Holm equation.

Starting from the Green-Naghdi model of shallow water theory, we derive a system of partial differential equations which can be considered as a generalization to two space dimensions of the equation derived by Camassa and Holm to model the dynamics of water waves.

Solutions of our system of differential equations correspond to geodesics on the group of diffeomorphisms of $\mathbf{R}^2$ with respect to a certain right invariant metric on this group. Solutions of the Camassa-Holm equation can be interpreted as solutions to the new equations, that depend on only one of the two space variables.

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Mohammed Reza Molaei

Bifurcation Points Created By A Class of One-Parameter Families of Maps.

In this talk we will consider a class of one-parameter families of maps with critical points. We will deduce a kind of bifurcation points by considering Lyapunov exponents.

James Montaldi

Group theoretic criteria for robust homoclinic connections

Beginning with the well-known example of Guckenheimer and Holmes, a number of examples of equivariant systems have been found that exhibit robust homoclinic connections. In this talk I will describe the unifying group-theoretic structures that permit this phenomenon. In particular, one immediately sees that such robust homoclinic connections do not occur for Abelian groups, and also that the only dihedral groups allowing such connections are those of order an integer multiple of 8. [Joint work with Peter Ashwin.]

M. Concepcion Muriel and J.L. Romero

Ordinary differential equations and non-solvable symmetry algebras

We have defined a new class of symmetries ($C^\infty-$symmetries) for ordinary differential equations. They have associated an algorithm to reduce the order of the equation, which is more general than the one corresponding to Lie symmetries.

In this work we apply the reduction process based on $C^\infty-$symmetries to ordinary differential equations whose symmetry algebra is non-solvable. The method provides a step by step process of reduction, and, as a consequence, the general solution of the original equation can be obtained from the reduced equation by solving first order differential equations.

A classification of third order equations that admit three-dimensional non-solvable symmetry algebra is given, as well as the corresponding first order reduced equations. We also show how the general solution of the original equation can be obtained from the solution of the reduced equation by means of two successive quadratures.

Nikolai Nekhoroshev

Generalizations of Gordon's theorem.

Gordon's theorem claims that given a Hamiltonian system all of whose solutions are periodic, the period of a solution depend only on the value of the Hamiltonian function on the trajectory of this solution. Generalizations are obtained for the case of invariant isotropic tori of arbitrary dimension k (rather than $k = 1$), which fiber either all the phase space or some submanifold of this. One supposes that the system has some collection of k first integrals in involution, such that the corresponding vector fields are tangent to these tori. Then the frequencies of quasiperiodic motion on such a torus depend only on the values of these first integrals on the torus. This is true also for the circular action functions, but sufficient conditions are essentially different in these two cases.

Juan-Pablo Ortega

Critical point theory and Hamiltonian dynamics around critical elements.

In this talk we will show how standard techniques in critical point theory can be adapted to the symplectic category in order to obtain various motions or solutions associated to Hamiltonian dynamical systems. Special emphasis will be given to systems with symmetries.

Jesús Palacián and Patricia Yanguas

Computing invariant manifolds of perturbed dynamical systems

A method to calculate invariant manifolds of perturbed dynamical systems defined through an m–dimensional vector field is presented. The technique is based on the calculation of formal symmetries and generalized normal forms through Lie transformations for ordinary differential equations. Once a symmetry is determined up to a certain order, a reduction map allows us to pass to the so–called reduced system, which is defined in the orbit space. Its dimension is $s < m$ and it is parametrized by the invariants associated to the reduction map. Now, a non–degenerate p–dimensional (with $p \leq s$) invariant set of the reduced system is transformed, asymptotically, into a $(p + m - s)$–dimensional invariant set of the departure system. Moreover, these invariant sets are calculated explicitly (up to an adequate order) by inverting the Lie transformations.

As an illustration, the procedure is applied to the Hamiltonian vector field defined by the restricted three body problem and to the Lorenz system, so as to characterize new periodic orbits and $2D$ invariant manifolds.

A. Delshams and J. Tomás Lázaro

Pseudo-normal forms and their applications

Pseudo-normal forms, which originally come from J. Moser and D. DeLatte, appear to be a useful generalization of the classical (Hamiltonian) Birkhoff normal forms. This new approach provides nice results for a wide range of systems: conservative, reversible and dissipative. Some of these applications will be presented in this talk, like (a) an approach to the (planar) *center-focus problem*; (b) the study of a general system in a neighborhood of a four-dimensional saddle-center equilibrium point; and (c) the attainment of good parameterizations for the stable and unstable whiskers corresponding to a hyperbolic invariant torus, problem which arises in the study of the splitting of separatrices of a reversible system under the action of a quasi-periodic reversible perturbation.

Tudor Ratiu

A review of Lagrangian reduction

This talk will present some of the key ideas in Lagrangian reduction theory. It will begin with the Euler Poincare equations and show how these are generalized to equations and variational principles for quotients of tangent bundles by Lie group symmetries.

Vladimir Rosenhaus

1-Dimensional infinite symmetries, boundary conditions and conservation laws

We will consider equations of dimension two and higher whose symmetry group generators contain an arbitrary function of one variable and its derivative. We will discuss the relationship between symmetries, boundary conditions, and local conservation laws. As an example we will study the equation of nonstationary transonic gas flow. Comparison with the Second Noether Theorem will be discussed.

Dmitrii Sadovskii

Normal forms, geometry, and dynamics of atomic and molecular systems with symmetry.

We show how general ideas of Michel, Zhilinskii, Zak, and others on the application of topology, group theory, and commutative algebra in the analysis of symmetric dynamical systems can be concretized for Hamiltonian dynamical systems with symmetries. Particular examples include perturbed Kepler systems (such as hydrogen atom in external fields) and molecular rotation-vibration systems. We consider reduction of continuous dynamical symmetries using normal forms and study singular reduction, geometry, and Poisson structure of reduced systems using polynomial invariants. Particular attention is paid to further analysis of residual continuous and discrete symmetries of the reduced system and to characterization of relative equilibria.

Jan A. Sanders

Integrable systems and Riemannian manifolds with constant curvature

We look at arc-lenght preserving evolution of a curve on a 3-dimensional Riemannian manifold. Using the moving frame method it follows from the structural equations and integrability conditions that the evolution of the curvature and torsion (of the curve) is determined by a Hamiltonian operator, which splits into three compatible Hamiltonian operators, giving rise to a number of integrable systems. From these we obtain two integrable geometric evolution equations for the curve. These equations are parametrized by the curvature of the manifold. An interesting new feature is that we find, after a hodograph transform, a symmetry among the operators that disappears when the curvature of the manifold goes to zero. [Joint work with Gloria Mari Beffa and Jing Ping Wang.]

Dominique Sugny and Marc Joyeux

Recent advances in canonical perturbation theory, with application to chemical reactions and nonadiabatic dynamics.

Canonical Perturbation Theory (CPT) is a powerful tool in the field of molecular physics. It consists of a series of coordinate transformations, which are aimed at rewriting the Hamiltonian in a simpler form (that is, in terms of as many good quantum numbers as possible) without modifying the geometry of the phase space. The major achievement of CPT is the straightforward derivation of relations between the physically meaningful parameters of potential energy surfaces and the coefficients of the so-called effective Hamiltonians. Until recently, however, explicit high order calculation schemes have been known only for motion around a single minimum, so that only rigid molecules moving on an uncoupled Born-Oppenheimer electronic surface could be investigated.

The first goal of this talk is to report on some recent advances, which allow for the successful application of CPT to "floppy" systems with several equilibrium positions [D. Sugny and M. Joyeux, *J. Chem. Phys.* **112**, 31 (2000); D. Sugny, M. Joyeux and E.L. Sibert, *J. Chem. Phys.* **113**, 7165 (2000)] : The vibrational modes are sorted into two categories, namely oscillator-like ones and hindered-rotor-like ones. The application of CPT enables to express the Hamiltonian in terms of the good quantum numbers and/or classical constants of the motion associated with the oscillator-like modes. The perturbative Hamiltonian then acts on the reduced dimensional space of the hindered-rotor-like modes. The validity and accuracy of this approach has been tested on the simplest chemical reaction, namely the $HCN \leftrightarrow CNH$ isomerization reaction, with some unexpected results regarding its dynamics.

In the second part of the talk we propose another new CPT procedure, based on Van Vleck's theory, for investigating nonadiabatic (i.e. non Born-Oppenheimer) dynamics. The basic idea is to let the unitary transformations depend on the matrix representations of angular momentum operators. This procedure has been applied to a simple model with two 2-dimensional diabatic electronic surfaces coupled by a ℓq term, which mimicks the bend and antisymmetric stretch degrees of freedom of $NO2$ in the ground and first excited electronic states. The anharmonicities induced in the lowest surface by the coupling to the excited one are very precisely computed by the proposed procedure.

Susanna Terracini

Nehari problem in two dimensions and competing species systems

We shall talk about the asymptotic behaviour of solutions of variational elliptic systems, as the coefficient of the interaction term tends to infinity. An analogous problem, in a non variational setting, has been faced by Dancer & Du, in the context of the study of competing species with diffusion; Dancer & Du showed a link between the asymptotic behaviour of the solutions to the systems and the existence of sign-changing solutions to a suitable limiting single equation, and conversely. In this way, they obtained a deeper understanding of some analytic and geometric aspects of the segregation phenomenon. In the variational framework, the segregation problem seems more likely to be connected with an extension of Nehari's method [N] to the N dimensional case, through an approximation technique involving variational systems with competitive interactions. We shall prove the existence of solutions to the Nehari problem in dimension N in the case of two nodal regions. Afterwards, goal we'll analyze the situation of an arbitrary number of nodal domains and study the geometrical properties of the free boundary. To deal with the geometrical properties of the common boundaries, we will use the theoretical tools that have been developed by several authors in the field of two phase and other free boundary problems. These techniques involve tools from the Calculus of Variations, Geometric Measure Theory and Real Analysis.

André Vanderbauwhede

On the Continuation of Periodic Orbits in Symmetric Hamiltonian Systems

In this talk we discuss a particular method for the continuation (both theoretically and numerically) of periodic orbits in Hamiltonian systems having a number of constants of motion which may or may not be in involution. In such systems periodic orbits belong to multi-parameter families of such orbits, and the multiplier 1 will have a higher multiplicity; we give lower estimates for both the geometric and the algebraic multiplicity of this multiplier. Our continuation method extends and generalizes a classical technique which embeds Hamiltonian systems in a 1-parameter family of (dissipative) systems by adding a multiple of the gradient of the Hamiltonian to the vectorfield. We apply our results to some simple examples which are chosen such that they also give a more geometrical insight into the method. [Talk based on joint work with F. J. Muñoz-Almaraz, E. Freire, E. Doedel and J. Galán]

Jing Ping Wang

Classification of integrable systems with number theoretic methods.

The classification of evolution equations with respect to the existence of (infinitely many) symmetries is a difficult problem, on which several groups have been working for years now. Recently there has been a breakthrough by the use of number theoretic methods, most notably diophantine approximation theory and p-adic analysis. In this talk I will give a survey of the results of this approach. [Joint work with Jan Sanders.]

Claudia Wulff

Stability of Hamiltonian relative equilibria

For a compact group G a relative equilibrium of a G-invariant Hamiltonian is orbitally Liapounov stable (G-stable) if the Hessian of an augmented Hamiltonian at the corresponding critical point is definite when restricted to the symplectic normal space. This is no longer true in general for proper actions of noncompact groups, essentially because the orbit space of the coadjoint action of the group need not be Hausdorff. This talk will present a more general stability criterion that applies to relative equilibria with "tame" angular velocities, and also give a result that sharpens G-stability to A-stability, where A is a subset of G that depends primarily on the momentum of the relative equilibrium. [Joint work with G. Patrick and M. Roberts]

Joshua Zak

Topologically unavoidable degeneracy in band structure of solids

Rules are developed for a new kind of topologically unavoidable branch crossing in band structure of solids. It is proven that there exist a great variety of 4-branch bands in crystalline solids of the orthorhombic system in which the crossings are necessitated by symmetry and topology. These crossings can take place on either points or lines in the Brillouin zone.