ON AUTOMATON RECOGNIZABILITY OF ABNORMAL EXTREMALS

UGO BOSCAIN* AND BENEDETTO PICCOLI†

Abstract. For a generic single input planar control system $\dot{x} = F(x) + uG(x)$, $x \in \mathbb{R}^2$, $u \in [-1, 1]$, $F(0) = 0$, we analyze the properties of abnormal extremals for the minimum time stabilization to the origin. We prove that abnormal extremals are finite concatenations of bang arcs with switchings occurring on the set in which the vector fields F and G are collinear. Moreover, all the generic singularities of one parametric families of extremal trajectories near to abnormal extremals are studied. In particular we prove that all possible sequences of these singularities, and hence all generic abnormal extremals, can be classified by a set of words recognizable by an automaton.

Key words. Optimal control, Abnormal extremals, Synthesis theory, Generic planar systems

AMS subject classifications. 49k15, 49j15, 49n35

1. Introduction. In this paper we deal with the minimum time stabilization problem to the origin for the planar single-input system $\dot{x} = F(x) + uG(x)$, $x \in \mathbb{R}^2$, $u \in [-1, 1]$, $F(0) = 0$. The Pontryagin Maximum Principle (in the following PMP) [11, 18] provides a necessary condition for optimality and trajectories satisfying it are called extremals. Abnormal extremals are extremals corresponding to the zero level of the Hamiltonian given by the PMP, see [1, 2].

The aim of this paper is to analyze completely the generic properties of abnormal extremals. We prove that these are finite concatenations of bang arcs (that is corresponding to constant ± 1 control). Moreover, the switchings (discontinuity points of the control) happen exactly when the abnormal extremal crosses the set of zeroes of the function $\Delta_A = F \wedge G$. In many cases, an abnormal extremal is formed by fold points, that is there are two sheets of the cotangent bundle covered by extremal trajectories that project onto the same region of the plane and the boundary of this region is the projection of the abnormal extremal.

The set of possible singularities along abnormal extremals is formed of 28 (equivalence classes of) singular points, but not all sequences of singularities can be realized. We aim to provide a classification of all possible sequences of singularities. A good classification is obtained if one can put the possible sequences in bijective correspondence with some algebraic or combinatorial object Ω with simple structure. If all possible sequences of singularities were admitted, then this classification can be done choosing Ω to be the set of all words formed with letters from the alphabet $\{1, \ldots, 28\}$, with the meaning that each number corresponds to a singularity. This is not the case, however we can still have some regular structure, more precisely Ω can be chosen as a set of words recognizable by an automaton and this is the most natural classification for this problem, see rules **R1**,**R2**,**R3** of Section 5. We recall that a set of words Ω from a given finite alphabet is recognizable if there exists an automaton that generates exactly the words of Ω. An automaton is roughly speaking a graph, with labelled edges, and one constructs words considering all paths starting from a given set of points and ending to another fixed set of points. The recognizable sets of words share a regular structure used in theoretical computer science, see [10]. Since the sequence

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of singularities completely describes the abnormal extremal, we obtain that abnormal extremals are recognizable by an automaton.

In [14, 15, 21, 22], the authors studied the properties of extremal trajectories and via a finite dimensional reduction, obtained using the Pontryagin Maximum Principle, proved the existence of a regular optimal synthesis. Roughly speaking, an optimal synthesis is a function associating to each point an optimal trajectory. It happens that the synthesis is indeed generated by a feedback that is smooth on each strata of a Withney stratification of the plane. For syntheses theory we refer also to [3, 9, 17]. In a series of papers [7, 8, 15, 16], the existence of a structurally stable optimal synthesis for generic smooth systems was proved and complete classifications of the corresponding (non smooth) flows and relative singularities were given, in the same spirit of the work of Peixoto for two dimensional dynamical systems. For results in three dimension, see [19, 20]

An alternative approach is used here and in [4, 5, 12, 13]. This amounts to constructing all extremals, that is trajectory-covector pairs satisfying the PMP, and projecting the obtained set onto the plane. This approach is more involved but shades more light on the links between synthesis singularities and the singularities of the minimum time function, see [6]. Moreover, the supports of extremals form a Withney stratified set of dimension three in the cotangent bundle (see [5]). After normalization of the covector, one obtains a two dimensional stratified set in $\mathbb{R}^2 \times S^1$. The projection singularities can be classified in topological sense. Beside the classical folds and cusps, see [23], new singularities appear. Some, called vertical, are due to the fact that the target (the origin) is of codimension two, while others are stable and independent of the target properties. These new singularities are called bifold and ribbon. In particular the ribbon singularity can appear only along abnormal extremals.

All possible generic singularities of the synthesis on the plane occurring along (projections of) abnormal extremals are classified in Section 4. However, one has to prove that all singularities indeed appear for some generic system, in particular those corresponding to the new projection singularities, namely bifold and ribbon. As a byproduct, we obtain the existence of systems presenting singular points corresponding to projection singularities of bifold and ribbon type (see Theorem 5.5).

The paper is organized in the following way. Section 2 introduces basic definitions and main results. The third Section is dedicated to formulate and prove the main Propositions about the switching strategy of abnormal extremals. In Section 4 we describe the synthesis singularities (on the plane) occurring along abnormal extremals. Finally, in Section 5 we describe the classification of abnormal extremals via a recognizable set of words and the corresponding automaton. Moreover in Section 5 we show that the ribbon singularity is realized and we give an example of a synthesis containing a singularity of this kind.

2. Basic Definitions and Statement of the Main Result. Let Ξ be the set of all couples of $\mathcal{C}^\infty$ vector fields (F,G) such that the origin is an equilibrium point for F, that is $F(0)=0$. From now on we endow Ξ with the $\mathcal{C}^3$ topology induced by the norm:

$$\|(F,G)\| = \sup\Big\{ \left|\frac{\partial^{\alpha_1+\alpha_2}F_i(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right|, \left|\frac{\partial^{\alpha_1+\alpha_2}G_i(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right| : \\ x\in\mathbb{R}^2;\ i=1,2;\ \alpha_1,\alpha_2\in\mathbf{N}\cup\{0\};\alpha_1+\alpha_2\leq 3\Big\}.$$

We say that a subset of Ξ is generic if it contains an open and dense set. Analogously, a property P is said generic if the set satisfying P is generic.

For every $(F, G) \in \Xi$ we consider the minimum time stabilization problem to the origin for the control system:

$$\dot{x} = F(x) + uG(x), \qquad x \in \mathbb{R}^2, \quad u \in [-1, 1]. \tag{2.1}$$

Reversing time, we deal with the equivalent problem of reaching every point of the plane in minimum time from the origin.

Given a measurable function $u : [a, b] \to [-1, 1]$, a trajectory of (2.1) corresponding to u is an absolutely continuous map $\gamma : [a, b] \to \mathbb{R}^2$ such that $\dot{\gamma}(t) = F(\gamma(t)) + u(t)G(\gamma(t))$ for almost every $t \in [a, b]$. Since the system is autonomous we can always assume that $Dom(\gamma) = [0, a]$ for some $a \in \mathbb{R}$, $a > 0$, where Dom denotes the domain. Moreover we denote by $Supp(\gamma)$ the set $\gamma([a, b])$. A trajectory $\gamma : [0, a] \to \mathbb{R}^2$ is (time) optimal if for every trajectory $\gamma' : [0, b] \to \mathbb{R}^2$ with $\gamma(a) = \gamma'(b)$ we have $a \leq b$. A trajectory γ corresponding to a constant control ± 1 is called a *bang arc*. A *bang-bang* trajectory is a finite concatenation of bang arcs and a time where the control changes sign is called *switching time*.

The well known Pontryagin Maximum Principle in this special case states the following. Define for every $(x, p, u) \in \mathbb{R}^2 \times (\mathbb{R}^2)_* \times [-1, 1]$, where $(\mathbb{R}^2)_*$ is the set of row vectors,

$$\mathcal{H}(x, p, u) = p \cdot F(x) + u\, p \cdot G(x) \tag{2.2}$$

and:

$$H(x, p) = \max\{p \cdot F(x) + u\, p \cdot G(x) : u \in [-1, 1]\}. \tag{2.3}$$

If $\gamma : [0, a] \to \mathbb{R}^2$ is a (time) optimal trajectory corresponding to a control $u : [0, a] \to [-1, 1]$, then there exist a non trivial *field of covectors along* γ, that is a function $\lambda : [0, a] \to (\mathbb{R}^2)_*$ never vanishing, and a constant $\lambda_0 \leq 0$ such that:

i) $\dot{\lambda}(t) = -\lambda(t) \cdot (\nabla F + u(t)\nabla G)(\gamma(t))$,

ii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$ for almost every $t \in Dom(\gamma)$,

iii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t))$ for almost every $t \in Dom(\gamma)$.

In this case we say that the pair (γ, λ) is extremal. If γ is optimal we say that the pair (γ, λ) is optimal.

Given an extremal pair (γ, λ) one easily checks that for every $\alpha \in \mathbb{R}$, $\alpha > 0$, the pair $(\gamma, \alpha\lambda)$ is also extremal.

Definition 2.1. *Let (γ, λ) be an extremal pair. If the corresponding Hamiltonian satisfies $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = 0$ for almost every $t \in Dom(\gamma)$ (i.e. if λ_0 in (i) above vanishes) we say that (γ, λ) is an abnormal extremal.*

The following Theorem, proved in Section 5, describes the set of generic abnormal extremals.

Theorem 2.2. *The set of abnormal extremals for control systems (2.1), in a generic set of Ξ, can be classified through a set of words recognizable by an automaton.*

In the following we introduce some notations also used in [7, 15, 16]. Set:

$$Y = F + G, \qquad X = F - G.$$

Let $\gamma : [t_1, t_2] \to \mathbb{R}^2$ be a trajectory of (2.1). If γ corresponds to constant control +1 (resp. -1) on $[t_1, t_2]$ we say that $\gamma|_{[t_1,t_2]}$ is a Y–trajectory (resp. X–trajectory) and we write $\gamma|_{[t_1,t_2]} \in$ Traj(Y) (resp. $\gamma|_{[t_1,t_2]} \in$ Traj(X)). We say that $\gamma|_{[t_1,t_2]}$ is a

Z–trajectory and we write $\gamma|_{[t_1,t_2]}$ $\in$Traj(Z), if γ on $[t_1,t_2]$ corresponds to the control (called singular):

$$\varphi(x) = -\frac{\nabla\Delta_B(x)\cdot F(x)}{\nabla\Delta_B(x)\cdot G(x)}, \tag{2.4}$$

where:

$$\Delta_B(x) = G(x)\wedge[F,G](x) = G_1(x)[F,G]_2(x) - G_2(x)[F,G]_1(x),$$

and $[F,G]$ is the Lie bracket of F and G. For later use we also define the function:

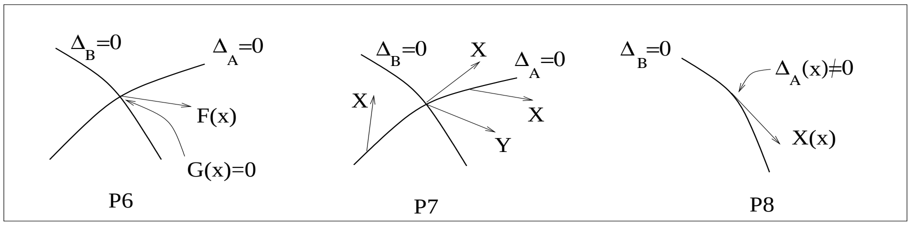
$$\Delta_A(x) = F(x)\wedge G(x) = F_1(x)G_2(x) - F_2(x)G_1(x).$$

Definition 2.3. *If $\gamma_1 : [t_1,t_2] \to \mathbb{R}^2$, $\gamma_2 : [t_2,t_3] \to \mathbb{R}^2$ $(t_1 < t_2 < t_3)$ are two trajectories of (2.1) with $\gamma_1(t_2) = \gamma_2(t_2)$, we set $(\gamma_2 * \gamma_1)(t) := \gamma_1(t)$ if $t \in\ [t_1,t_2[$ and $(\gamma_2 * \gamma_1)(t) := \gamma_2(t)$ if $t \in\ [t_2,t_3]$. Given an extremal trajectory γ we denote by $n(\gamma)$ the smallest integer such that there exist $\gamma_i \in$ Traj(X)$\cup$Traj(Y) $\cup$Traj(Z), $i = 1,...,n(\gamma)$, satisfying $\gamma = \gamma_{n(\gamma)} * ... * \gamma_1$. The function $n(\gamma)$ is called the number of arcs of γ.*

In the following, we assume the generic conditions:

(P1) The vectors $G(0)$ and $[F,G](0)$ are linearly independent.
(P2) Zero is a regular value for Δ_A and Δ_B.
(P3) The set $\Delta_A^{-1}(0)\cap\Delta_B^{-1}(0)$ is locally finite.
Let Tan_A be the set of points $x \in \Delta_A^{-1}(0)$ such that $X(x)$ or $Y(x)$ is tangent to $\Delta_A^{-1}(0)$. Define Tan_B in the same way using Δ_B rather than Δ_A.
(P4) Tan_A and Tan_B are locally finite sets.
Let $Bad := (\Delta_A^{-1}(0)\cap\Delta_B^{-1}(0))\cup Tan_A\cup Tan_B$.
(P5) Bad is locally finite.
Notice that **(P5)** is a consequence of **(P3)** and **(P4)**.
(P6) If $x \in Bad$, $G(x) = 0$, then $F(x)\cdot\nabla(\Delta_A)(x) \neq 0$ and $F(x)\cdot\nabla(\Delta_B)(x) \neq 0$.
(P7) If $x \in Bad$, $G(x) \neq 0$, $x \in (\Delta_A^{-1}(0)\cap\Delta_B^{-1}(0))\cap Tan_A$, say $X(x)\cdot\nabla\Delta_A(x) = 0$, then $\partial_y(X\cdot\nabla\Delta_A)|_{y=x} \neq 0$, $X(y) \neq 0$, $Y(y) \neq 0$, where y takes values on $\Delta_A^{-1}(0)$ in a neighborhood of x.
(P8) If $x \in Bad$, $x \in Tan_B$ then $\Delta_A(x) \neq 0$.
(P9) If $x \in Bad$, $G(x) \neq 0$, $X(x) = 0$ or $Y(x) = 0$ then $\Delta_B(x) \neq 0$.

The generic conditions **(P6)**, **(P7)**, **(P8)** are clarified in the following figure:



In [15] it was proved the following:

Proposition 2.4. *If* **(P1)**–**(P9)** *hold then each extremal trajectory γ can always be written in the form $\gamma = \gamma_{n(\gamma)} * ... * \gamma_1$ with $n(\gamma) < \infty$ and each arc is bang or singular (that is a Z–trajectory).*

Let $\tau > 0$. We call *reachable set* within time τ the set $\mathcal{R}(\tau) := \{x \in \mathbb{R}^2$: there exists $t \in [0,\tau]$ and a trajectory $\gamma : [0,\tau] \to \mathbb{R}^2$ of (2.1) such that $\gamma(0) = 0$, $\gamma(t) = x\}$. In [15] it was shown:

LEMMA 2.5. *Under the generic conditions* **(P1)–(P9)** $\gamma_i \in$ *Traj(Z) iff Supp*$(\gamma_i) \subset \Delta_B^{-1}(0)$.

In the case in which $\gamma_i \in \mathrm{Traj}(Z)$ we say that $Supp(\gamma_i)$ is a *turnpike*. We call $\gamma^\pm : [0, t_f^\pm] \to \mathbb{R}^2$ the extremal trajectories originate at 0 and correspond to constant control ± 1 with $t_f^\pm$ the last times in which $\gamma^\pm$ are extremal (if they are less than τ) or τ (otherwise). Under generic assumptions, every extremal trajectory exits the origin with constant control $+1$ or -1.

DEFINITION 2.6. *Let* $(\gamma, \lambda) : [0,\tau] \to \mathbb{R}^2$ *be an extremal pair. The corresponding switching function is defined as* $\phi(t) := \lambda(t) \cdot G(\gamma(t))$.

We have immediately the following:

LEMMA 2.7. *Let* $(\gamma, \lambda) : [0,\tau] \to \mathbb{R}^2$ *be an extremal pair and* $[t_1, t_2] \subseteq [0,\tau]$ $(t_1 < t_2)$. *Then:*

- *on* $[t_1,t_2]$, γ *corresponds to constant control +1 (resp. -1) iff for each* $t \in [t_1,t_2]$ *we have* $\phi(t) \geq 0$ *(resp.* $\phi(t) \leq 0$*) and meas*$(\{t \in [t_1,t_2] : \phi(t) = 0\}) = 0$;
- *on* $[t_1,t_2]$, γ *corresponds to the singular control* φ *(defined in (2.4)) iff for each* $t \in [t_1,t_2]$ *we have* $\phi(t) = 0$.

In [21] it was proved the following:

LEMMA 2.8. *Let* γ *be an extremal trajectory and* $\bar{t} \in Dom(\gamma)$ *a switching time. Let* $\bar{x} = \gamma(\bar{t})$ *and suppose that* $\bar{x} \notin \Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$. *Then* $\bar{t}$ *is a switching time from* X *to* Y *iff* $-\Delta_A(\bar{x})/\Delta_B(\bar{x}) > 0$ *and* $\bar{t}$ *is a switching time from* Y *to* X *iff* $-\Delta_A(\bar{x})/\Delta_B(\bar{x}) < 0$.

We now introduce some definitions to describe the projection singularities we encounter along abnormal extremals. Fix $\bar{x} \in \mathcal{R}(\tau)$ and an extremal trajectory $\bar{\gamma} : [0,\bar{a}] \to \mathcal{R}(\tau)$ such that $\bar{\gamma}(0) = 0$, $\bar{\gamma}(\bar{a}) = \bar{x}$ and define the function:

$$K_\varepsilon^{\bar{x},\bar{\gamma}}(x) := \sharp\{\text{extremal trajectory } \gamma : \ \gamma(0) = 0,\ \gamma(a) = x,\ |\gamma(t) - \bar{\gamma}(t)| < \varepsilon \quad \forall\, t \in [0, min(a,\bar{a})],\ |a - \bar{a}| < \varepsilon\}, \tag{2.5}$$

where $\sharp$ denotes the cardinality of a set. Referring to Figure 2.1 we introduce the following key definition:

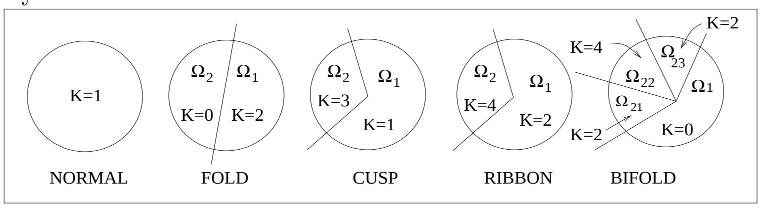


Fig. 2.1

DEFINITION 2.9. *Fix* $\bar{x} \in \mathcal{R}(\tau)$ *and an extremal trajectory* $\bar{\gamma} : [0,\bar{a}] \to \mathcal{R}(\tau)$ *such that* $\bar{\gamma}(0) = 0$ *and* $\bar{\gamma}(\bar{a}) = \bar{x}$.

- *We say that* $\bar{x}$ *is a* normal point *along* $\bar{\gamma}$ *if for* ε *sufficiently small there exists a neighborhood* U *of* $\bar{x}$ *such that* $K_\varepsilon^{\bar{x},\bar{\gamma}}(U) = 1$.
- *We say that* $\bar{x}$ *is a* fold point *along* $\bar{\gamma}$ *if there exists a one dimensional piecewise-*$\mathcal{C}^1$ *manifold* l, *with* $\bar{x} \in l$, *satisfying the following. For* ε *sufficiently small there exists a neighborhood* U *of* $\bar{x}$ *divided by* l *into two connected components* Ω_1, Ω_2 *such that* $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1) = 2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_2) = 0$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(l) = 1$.

- *If l, $U(\varepsilon)$, Ω_1, Ω_2 are as above then if $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1)=1$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_2)=3$ and $K_\varepsilon^{\bar{x},\bar{\gamma}}(l)=2$, we say that $\bar{x}$ is a cusp point along $\bar{\gamma}$.*
- *If l, $U(\varepsilon)$, Ω_1, Ω_2 are as above then if $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1)=2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_2)=4$ and $K_\varepsilon^{\bar{x},\bar{\gamma}}(l)=3$, we say that $\bar{x}$ is a ribbon point along $\bar{\gamma}$.*
- *We say that $\bar{x}$ is a bifold point along $\bar{\gamma}$ if there exists a one dimensional connected piecewise-$\mathcal{C}^1$ embedded manifold l and two connected $\mathcal{C}^1$ embedded manifolds l_1 and l_2 satisfying the following:*
  1. *$l\cap\partial l_i=\{\bar{x}\}$ $(i=1,2)$; $\partial l_1\cap\partial l_2=\{\bar{x}\}$;*
  2. *for ε sufficiently small there exists a neighborhood U of x satisfying the following. The set $U\setminus l$ has two connected components Ω_1, Ω_2 and the set $\Omega_2\setminus\{l_1\cup l_2\}$ has three connected components Ω_{21}, Ω_{22}, and Ω_{23} (the name are chosen as in fig. 2.1) such that $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1)=0$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_{21})=2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_{22})=4$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_{23})=2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(l)=1$ and $K_\varepsilon^{\bar{x},\bar{\gamma}}(l_i)=3$ $(i=1,2)$.*

Notice that this definition describes projection singularities. Indeed the trajectories reaching a same point in the plane correspond to different lifts in the cotangent bundle.

Our aim is to prove that these are the only projection singularities that may happen along abnormal extremals and show that they are in fact realized for some control systems.

The behavior of abnormal extremals is individuated by the singularities it meets. To describe all possible singularities, we start describing those encountered in the optimal case. In [7, 15, 16], dealing with the optimal synthesis, it was proved that $\mathcal{R}(\tau)$ is a stratified subset of $\mathbb{R}^2$ and the one and zero dimensional strata that are singularities of the optimal flow, are called respectively *Frame Curves* and *Frame Points* (in the following briefly FCs and FPs). Moreover, the authors use the letters X, Y, C, S, K, to indicate respectively the FCs corresponding to subsets of $Supp(\gamma^+)$, subsets of $Supp(\gamma^-)$, curve made of switching points (switching curves, see Figure 2.2 case 1), turnpikes and overlaps, that is curves made of points reached optimally by two distinct trajectories. In [16], it was proved that these are generically all possible FCs. A FP x that is intersection of two FCs F_1 and F_2 is called a (F_1,F_2) Frame Point. Considering the extremal trajectories instead of the optimal ones we do not have any K Frame Curve but we have the following new Frame Curves:

- a FC called $\bar{C}$ made of switching points on which X and Y points to opposite sides (see Figure 2.2, case 2);
- a FC called W that is an arc of an extremal trajectory characterized by the fact that all its points are fold points (see Figure 2.2, case 3).
- A FC called γ_0 that is an arc of an extremal trajectory that "transports" some special information e.g. it switches every time it meets the locus $\Delta_A^{-1}(0)$ or it evolves into a W FC.

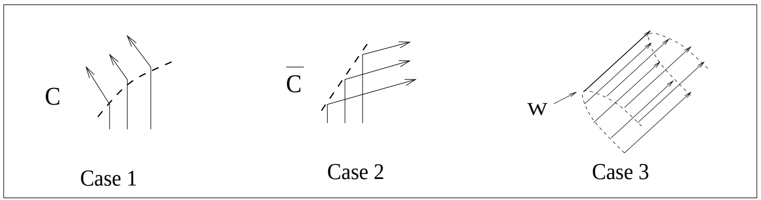


Fig. 2.2. *the FCs C, $\bar{C}$, W*

More details on the Frame Curves W and γ^0 are given below.

To understand in details the structure of abnormal extremals, we describe the set of all extremals, that is organized in one parameter families called strips. Let Γ be the set of all the extremal trajectories for the control problem (2.1) up to time $\tau > 0$. Again, under generic assumptions, extremals are finite concatenations of bang and singular arcs, so Γ can be obtained by a constructive algorithm that makes inductive steps on the number of arcs of extremal trajectories. We outline such algorithm, described in details in [5].

Outline of the Algorithm. First one constructs all the extremal trajectories in a neighborhood of $\gamma^{\pm}$. Then the set of extremal trajectories is subdivided into "strips". Each strip is a one dimensional continuous parametric family of extremal bang–bang trajectories having the same switching strategy. The evolution of each strip must be studied. In order to do that the evolution of the boundary of strips must be analyzed separately. The more delicate case is that of abnormal extremals that here we treat in details. The evolution of the interior of a strip may create new strips and cause the subdivision of each strip in smaller strips. This case is treated in [5], where the whole construction is completed.

Let us give the precise definition of strip.

DEFINITION 2.10. *Let a, b be two real numbers s.t. $0 \le a < b \le \tau$ and $x \in \mathcal{R}(\tau)$. A set of trajectories $\mathcal{S}^{a,b,x} = \{\gamma_\alpha : \ \alpha \in [a,b],\ \gamma_a(a) = x\}$ is called a strip if:*

i) $\forall\ \alpha \in [a,b],\ \ \gamma_\alpha : [0, \tau(\alpha)] \to \mathbb{R}^2$, $\tau(\alpha) > \alpha$ *is an extremal trajectory for the control problem (2.1). Moreover there exists $\varepsilon > 0$ s. t. $\gamma_{[\alpha,\alpha+\varepsilon]}$ corresponds to a constant control ± 1;*

ii) $\forall\ \alpha \in]a,b[,\ \ \gamma_\alpha$ *does not switch on $\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$ after time α;*

iii) *The set $\mathcal{B}^{a,b,x} = \{y \in \mathcal{R}(\tau) : \ \exists\ \alpha \in\ ]a,b[$ and $t \in]\alpha, \tau(\alpha)[$ s. t. $y = \gamma_\alpha(t)$, t is a switching time for $\gamma_\alpha\}$ is never tangent to X or Y.*

iv) *The map $\eta :\ \alpha \in\ [a,b] \mapsto \gamma_\alpha(\alpha) \in \mathbb{R}^2$ is a bang or singular arc and $\gamma_\alpha|_{[0,\alpha']} = \gamma_{\alpha'}|_{[0,\alpha']}$ if $a \le \alpha' \le \alpha \le b$.*

The function $\eta : [a,b] \to \mathbb{R}^2$ is called the *base* of the strip, $\overset{\circ}{\mathcal{S}}{}^{a,b,x} := \{\gamma_\alpha : \ \alpha \in]a,b[\}$ is called an open strip and $\partial\mathcal{S}^{a,b,x} := \{\gamma_a,\ \gamma_b\}$ is called *strip border*. See Figure 2.3 for a graphical description of a strip.

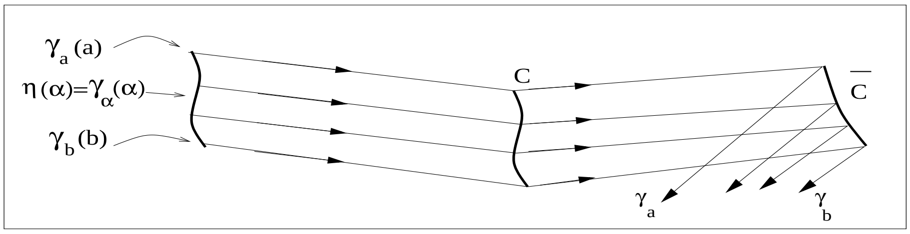


Fig. 2.3. *Strip.*

Notice that in [7, 15] a similar algorithm was used to construct the optimal synthesis that can be organized as a finite collection of optimal strips. Also in this case the algorithm constructs Γ as a finite union of strips. The set of borders of strips is called $\partial\Gamma$. We clearly have:

LEMMA 2.11. *Under generic assumptions, $\partial\Gamma$ is a finite set.*

Moreover, by construction, every border belongs to two different strips and adjacent strips correspond locally to the same control. More precisely it holds:

LEMMA 2.12. *Let $\gamma \in \partial\Gamma$, $\mathcal{S}^1$ and $\mathcal{S}^2$ be the two strips such that $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$ and $\bar{t}$ be a switching time for γ. Then the switching loci of $\mathcal{S}^1$ and $\mathcal{S}^2$, passing through $\gamma(\bar{t})$ correspond both to switchings from X to Y or both to switchings from Y to X.*

The algorithm produces the FCs in the following way:

- the FCs of kinds C and $\bar{C}$ lie in the interior of the strips;
- the borders of strips contain FCs of kind W, γ^0, X, Y;
- the bases of strips contain FCs of kind X, Y, S.

In fact the FCs of kind γ^0 and W are borders of strips. i.e.:

DEFINITION 2.13. *Let $\gamma \in \partial\Gamma$ and suppose that it corresponds to a constant control in the interval $]b,c[$ $(0 < b < c \leq \tau)$.*

- *We say that in $]b,c[$ γ is a strip border of kind W if for every $t \in\ ]b,c[$, $x := \gamma(t)$ is a fold point.*
- *Viceversa if for every $t \in\ ]b,c[$ $\gamma(t)$ is a normal point, we say that in $]b,c[$ γ is a strip border of kind γ_0.*

By construction:

LEMMA 2.14. *If $\gamma \in \partial\Gamma$, γ is of kind W in $]a,b[$ and of kind γ_0 in $]b,c[$ $(0 \leq a < b < c \leq \tau)$ or viceversa then b is a switching time for γ.*

In the next Section we see that in the case of abnormal extremals we need a more precise definition for FCs of kind W.

Given an extremal trajectory γ, let us define $v^\gamma(v_0, t_0; t)$ to be the solution to the Cauchy problem:

$$\left\{ \begin{array}{lcl} \dot{v^\gamma}(v_0,t_0;t) & = & (\nabla F + u(t)\nabla G)(\gamma(t)) \cdot v^\gamma(v_0,t_0;t) \\ v^\gamma(v_0,t_0;t_0) & = & v_0, \end{array} \right. \tag{2.6}$$

where $u(t)$ is the control corresponding to γ. In [15], denoting $\bar{v}^\gamma(t) := v^\gamma(G(\gamma(t)), t; 0)$ it was defined the function:

$$\theta^\gamma : \ Dom(\gamma) \to \ [-\pi, \pi], \qquad \theta^\gamma(t) := arg(\bar{v}^\gamma(0), \bar{v}^\gamma(t)), \tag{2.7}$$

where arg is the angle measured counterclockwise. We have the following:

LEMMA 2.15. *For fixed t_0, t, the map $f_{t_0,t} : v_0 \mapsto v^\gamma(v_0, t_0; t)$ is linear and injective. Moreover let $\gamma : [a,b] \to \mathbb{R}^2$ be an extremal trajectory of (2.1), corresponding to a constant control $\bar{u}$. Then for every $t, t_0 \in [a,b]$ it holds: $v^\gamma((F+\bar{u}G)(\gamma(t_0)), t_0; t) = (F+\bar{u}G)(\gamma(t))$.*

In the following we use the notation $\theta^\pm(t) := \theta^{\gamma^\pm}(t)$ and $v^\pm := v^{\gamma^\pm}$. From Lemma 2.15 we have for every $t \in [0, t_f^\pm]$: $v^\pm((F \pm G)(\gamma^\pm(t)), t; 0) = (F \pm G)(0) = \pm G(0)$. We consider the generic conditions:

(GC1) For every $t \in [0, t_f^+]$, $G(\gamma^+(t)) \neq 0$.
(GC2) $\dot{\theta}^+(0) \neq 0$, $\dot{\theta}^+(t_f^+) \neq 0$.
(GC3) If $\dot{\theta}^+(t) = 0$, then $\theta^+(t) \neq 0$, $\ddot{\theta}^+(t) \neq 0$.
(GC4) If $t \neq s$ and $\dot{\theta}^+(s) = \dot{\theta}^+(t) = 0$ then $\theta^+(s) \neq \theta^+(t)$.
(GC5) If $t_f^+ = \tau$ then $\max\{|\theta^+(t) - \theta^+(\tau)|, \ \ t \in [0,\tau]\} < \pi$.

In [7, 15] it was shown:

LEMMA 2.16. *We have the following:*

(A) $t_f^+ = \left\{ \begin{array}{l} \tau \ \textit{if}\ |\theta^+(a) - \theta^+(b)| < \pi \ \ \forall \ \ a, \ b \in \ [0,\tau]; \\ \min\{t \in [0,\tau] : \ \ |\theta^+(a) - \theta^+(b)| = \pi \ \textit{for some}\ a, \ b \in \ [0,t]\} \ \textit{otherwise}; \end{array} \right.$

(B) *let γ be an extremal trajectory, then for almost every $t \in Dom(\gamma)$ we have $sgn(\dot{\theta}^\gamma(t)) = sgn(\Delta_B(\gamma(t)))$;*

(C) *let γ be an extremal trajectory, then for each $t \in\ Dom(\gamma)$ the conditions*

$\phi(t) = \lambda(t) \cdot G(\gamma(t)) = 0$, $\dot{\theta}^{\gamma}(t) \neq 0$ *imply that* t *is a switching time for* γ.

We single out the following times:

$$s_1^+ := \min\{t \in]0, t_f^+] : \quad \theta^+(t) = 0, \dot{\theta}^+(0) < 0\},$$
$$s_1'^+ := \min\{t \in]0, t_f^+] : \quad \theta^+(t) = 0, \dot{\theta}^+(0) > 0\}.$$

Similarly the times $s_1^-, s_1'^-$ were defined. From now on we assume the generic condition

(GAτ) **i)** $\tau \notin \{s_i^{\pm}, s_i'^{\pm}\}$,
ii) let (γ, λ) be an extremal pairs. Then $\phi(\tau) = \lambda(\tau) \cdot G(\gamma(\tau)) = 0$ implies $\gamma(\tau) \notin \Delta_A^{-1}(0)$.

Observation 1. We can have three situations (cfr. Figure 2.4):
1) $|\theta^{\pm}(a) - \theta^{\pm}(b)| < \pi$ for every $a, b \in [0, \tau]$. In this case $t_f^{\pm} = \tau$ and $|\theta^{\pm}(t_f^{\pm})| < \pi$;
2) $|\theta^{\pm}(a) - \theta^{\pm}(b)| = \pi$ for some $a, b \in]0, \tau]$. In this case $|\theta^{\pm}(t_f^{\pm})| < \pi$, $\theta^{\pm}$ has a maximum or a minimum in $]0, t_f^{\pm}[$ and either $s_1^{\pm} \neq 0$ or $s_1'^{\pm} \neq 0$
3) $|\theta^{\pm}(t_f^{\pm})| = \pi$. In this case, $s_1^{\pm}$ and $s_1'^{\pm}$ are not defined and generically we get $t_f^{\pm} < \tau$.

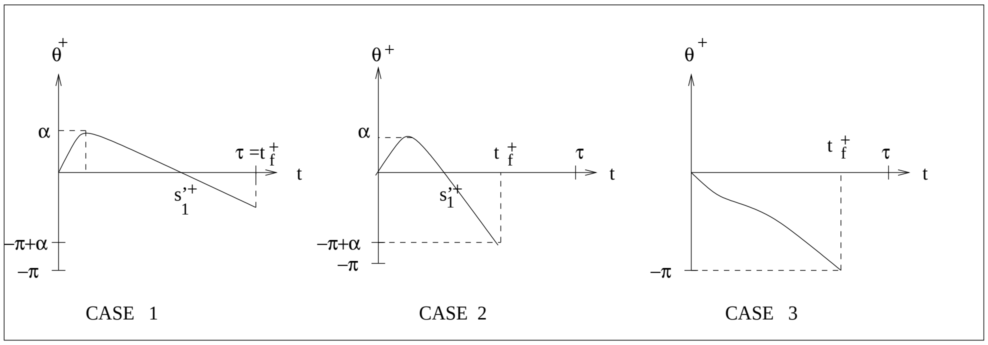


Fig. 2.4

3. Generic Properties of Abnormal Extremals. In this section we state and prove the main results about the switching strategies of abnormal extremals.

Definition 3.1. *Let* $\gamma : [0, \tau] \to \mathbb{R}^2$ *be (the first component of) an abnormal extremal for the control problem (2.1) such that it switches at least one time and let* t_1 *be its first switching time. We refer to the couples* (γ, t_1) *as Non Trivial Abnormal Extremal (in the following NTAE). By definition a NTAE is maximal if defined on* $[0, \tau]$.

Definition 3.2. *Let* $\gamma : [0, \tau] \to \mathbb{R}^2$ *be a NTAE, and* $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ *the sequence of switching times. We set* $AA(i) = Supp\left(\gamma|_{[t_i, t_{i+1}]}\right)$ *(*$i = 1, ..., n(\gamma) - 1$*) and we call it an abnormal arc.*

Proposition 3.3. *Let* $\gamma : [0, \tau] \to \mathbb{R}^2$ *be an extremal trajectory for the control problem (2.1) such that it switches at least one time,* $\lambda : [0, \tau] \to \mathbb{R}_2$ *the corresponding covector and* $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ *the sequence of switching times. Then it holds:*
A) $\lambda(.)$ *is unique (up to the multiplication by a positive constant);*
B) *under generic assumptions the following conditions are equivalent:*

(a) (γ, t_1) *is a NTAE;*
(b) $\gamma(t_i) \in \Delta_A^{-1}(0)$ *for some* $i \in \{1, ..., n(\gamma)-1\}$*;*
(c) $\gamma(t_i) \in \Delta_A^{-1}(0)$ *for each* $i \in \{1, ..., n(\gamma)-1\}$*;*
(d) $\gamma(\bar{t}) \in \Delta_A^{-1}(0)$ $(\bar{t} \in Dom(\gamma))$ *iff* $\bar{t} = t_i$ *for some* $i \in \{1, ...n(\gamma)-1\}$*;*

C) *under generic assumptions,* **a)** *(or equivalently* **(b)** *or* **(c)** *or* **(d)***) implies that for every interval* $[a, b] \subset [0, \tau]$ $(a < b)$*,* γ *does not correspond to the singular control* φ*.*

Proof. Proof of **A)**. The covector associated to an extremal trajectory is completely determined after the first switching. From $n(\gamma) \geq 2$ it follows that $\lambda(.)$ is unique up to a positive constant.
Proof that **(a)** implies **(c)**. Let (γ, λ) be an abnormal extremal such that $n(\gamma) \geq 2$. For each t_i $(i = 1, ..., n(\gamma)-1)$ we have $\lambda(t_i) \cdot G(\gamma(t_i)) = 0$ and there exists a sequence $t'_m \nearrow t_i$ such that

$$\mathcal{H}(\gamma(t'_m), \lambda(t'_m), u(t'_m)) = \lambda(t'_m) \cdot (F + u(t'_m)G)(\gamma(t'_m)) = 0.$$

Hence $\lambda(t'_m) \cdot F(\gamma(t'_m)) \to 0$ and $\lambda(t_i) \cdot F(\gamma(t_i)) = 0$. We can conclude that $F(\gamma(t_i))$ and $G(\gamma(t_i))$ are parallel, $\lambda(t_i)$ being not equal to 0. It follows **(c)**
Proof of **C)**. From **P3**, $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0) \cap \mathcal{R}(\tau)$ is a finite set. Generically γ does not switch on $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)$ Using Lemma 2.5 we conclude.
Proof that **(b)** implies **(a)**. Assume now that $\Delta_A(\gamma(t_i)) = 0$ for some $i \in \{1, ...n(\gamma)-1\}$. We have $\lambda(t_i) \cdot G(\gamma(t_i)) = 0$ and from $\Delta_A(\gamma(t_i)) = 0$ we get $\lambda(t_i) \cdot F(\gamma(t_i)) = 0$. There exists a sequence $t'_m \nearrow t_i$ such that:

$$\mathcal{H}(\gamma(t'_m), \lambda(t'_m), u(t'_m)) = \lambda(t'_m) \cdot (F + u(t'_m)G)(\gamma(t'_m)) \to 0.$$

From the PMP we know that $\mathcal{H}$ is almost everywhere equal to a fixed constant in $Dom(\gamma)$ hence we can conclude $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = 0$ almost everywhere.
Proof that **(a)** implies **(d)**. Fix $\bar{t}$ such that $\gamma(\bar{t}) \in \Delta_A^{-1}(0)$. We have $F(\gamma(\bar{t})) = \beta G(\gamma(\bar{t}))$ (by genericity we may assume $\beta \neq 0, \pm 1$) and there exists a sequence $t'_m \nearrow \bar{t}$ such that $|u(t'_m)| = 1$ and

$$\mathcal{H}(\gamma(t'_m), \lambda(t'_m), u(t'_m)) = \lambda(t'_m) \cdot (F + u(t'_m)G)(\gamma(t'_m)) = 0.$$

Hence $(1 + u(t'_m)\beta)\lambda(t'_m) \cdot G(t'_m) \to 0$ and being $\lim_{m\to\infty} u(t'_m) = \pm 1$ we have $\lambda(\bar{t}) \cdot G(\bar{t}) = 0$. From **(C)**, $\Delta_B(\gamma(\bar{t})) \neq 0$ thus $\dot{\theta}^{\gamma}(\bar{t}) \neq 0$ and $\bar{t}$ is a switching time (see Lemma 2.16). Viceversa, since **a)** implies **c)**, we get that $\Delta_A(\gamma(t_i)) = 0$ for each i. Thus we are done.
The implications **(c)⇒(b)**, **(d)⇒(b)** are obvious. This concludes the proof. □
We have:

LEMMA 3.4. *Let* (γ, t_1) *be a NTAE then* $\gamma \in \partial\Gamma$*.*

Proof. Let (γ, t_1) be a NTAE. From Proposition 3.3, $\gamma(t_1) \in \Delta_A^{-1}(0)$. Hence condition **ii)** of Definition 2.10 is violated, so γ can not be in the interior of a strip. □

Then it is natural to define $\partial\Gamma_A = \{\gamma \in \partial\Gamma,\ \gamma \text{ is an abnormal extremal}\}$.

DEFINITION 3.5. *Let* $x \in \Delta_A^{-1}(0)$ *be a switching point for a NTAE, then clearly* $X(x) \neq 0$*,* $Y(x) \neq 0$ *and* $(F+G)(x) = \alpha(F-G)(x)$ *for some* $\alpha \neq 0$*. If* $\alpha > 0$ *(resp.* $\alpha < 0$*) we say that at* x*,* $\Delta_A^{-1}(0)$ *is direct (resp. inverse).*

From Lemma 3.4 it follows that any $AA(i)$ is a strip border of kind γ_0 or W, but for abnormal extremals a more precise definition for the strip borders of kind W is necessary.

DEFINITION 3.6. *We refer to Figure 3.1. Let* $\gamma \in \partial\Gamma_A$*, and suppose that it corresponds to a constant control (say +1) in the interval* $]b,c[$ $(0 < b < c \leq \tau)$*. Let* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *be the two strips such that* $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$ *and suppose that in the interval* $]b,c[$ γ *is a strip border of kind* W*.*

- *We say that in* $]b,c[$ γ *is a strip border of kind* W^C *if* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *lie both on the right (resp. on the left) of* $\gamma|_{]b,c[}$ *and* X *points to the right (resp. to the left) of* $\gamma|_{]b,c[}$ *at every point of supp*$(\gamma|_{]b,c[})$*.*
- *We say that in* $]b,c[$ γ *is a strip border of kind* W^D *if* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *lie both on the right (resp. on the left) of* $\gamma|_{]b,c[}$ *and* X *points to the left (resp. to the right) of* $\gamma|_{]b,c[}$ *at every points of supp*$(\gamma|_{]b,c[})$*.*

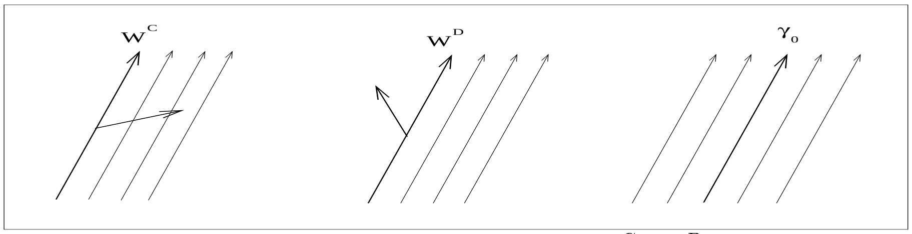


Fig. 3.1. *The abnormal extremals of type* W^C*,* W^D *and* γ_0*.*

From Proposition 3.3 we have the following:

LEMMA 3.7. *Let* $A_1, A_2 \in \{W^C, W^D, \gamma_0\}$ *and* $\gamma \in \partial\Gamma_A$*. If* γ *is of kind* A_1 *in* $]a,b[$ *and of kind* A_2 *in* $]b,c[$ $(A_1 \neq A_2,\ 0 < a < b < c \leq \tau)$ *then* b *is a switching time for* γ*.*

Now it is clear the meaning of this more fine definition in the case of abnormal extremal strip borders. An abnormal extremal can be of kind W^C (resp. W^D) on $[t-\varepsilon, t]$, $\varepsilon > 0$ and of kind W^D (resp. W^C) on $[t, t+\varepsilon]$, only if t is a switching time. On the contrary in the case of strip borders that are not abnormal extremals the change from W^C to W^D or viceversa can occur without switching and so the difference between W^C and W^D is not useful.

PROPOSITION 3.8. *Let* γ *be a NTAE,* $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ *be the sequence of its switching times and set* $t_0 = 0$*. From* **C**) *of Proposition 3.3, generically* $G(\gamma(t_i)) \neq 0$*. If* $F(\gamma(t_i)) = \beta_i G(\gamma(t_i))$*, then clearly* $\beta_i \neq \pm 1$ *(otherwise* $X(\gamma(t_i)) = 0$ *or* $Y(\gamma(t_i)) = 0$*.) For all* $i = 0, .., n(\gamma) - 2$ *it holds:*

$$v^\gamma(G(\gamma(t_{i+1})), t_{i+1}; 0) = (\beta_i + 1)/(\beta_{i+1} + 1) v^\gamma(G(\gamma(t_i)), t_i; 0)$$
$$\text{if } \gamma \text{ corresponds to control +1 on } [t_i, t_{i+1}]$$

$$v^\gamma(G(\gamma(t_{i+1})), t_{i+1}; 0) = (\beta_i - 1)/(\beta_{i+1} - 1) v^\gamma(G(\gamma(t_i)), t_i; 0)$$
$$\text{if } \gamma \text{ corresponds to control -1 on } [t_i, t_{i+1}]$$

Proof. Fix i and suppose that γ corresponds to constant control $+1$ in $[t_i, t_{i+1}]$, the opposite case being similar. From $F(\gamma(t_{i+1})) = \beta_{i+1} G(\gamma(t_{i+1}))$, recalling Lemma 2.15 we have:

$$\begin{aligned} F(\gamma(t_i)) + G(\gamma(t_i)) &= v^\gamma(F(\gamma(t_{i+1})) + G(\gamma(t_{i+1})), t_{i+1}; t_i) = \\ &= (1 + \beta_{i+1}) v^\gamma(G(\gamma(t_{i+1})), t_{i+1}; t_i). \end{aligned}$$

Now from $F(\gamma(t_i)) = \beta_i G(\gamma(t_i))$ (notice that in case $i = 0$ we have $F(0) = 0$, hence $\beta_0 = 0$) and using again Lemma 2.15 we have:

$$(1 + \beta_i) v^\gamma(G(\gamma(t_i)), t_i; 0) = (1 + \beta_{i+1}) v^\gamma(G(\gamma(t_{i+1})), t_{i+1}; 0)$$

that concludes the proof. □

PROPOSITION 3.9. *Let $\gamma : [0,\tau] \to \mathbf{R}^2$ be an extremal trajectory for the control problem (2.1) that switches at least one time, θ^γ the corresponding function defined in (2.7) and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_n(\gamma) := \tau$ the sequence of switching times. Then under generic assumptions the following conditions are equivalent:*

(a) (γ, t_1) *is a NTAE;*

(b) $\theta^\gamma(t_i) \in \{0, \pm\pi\}$ *for some* $i \in \{1, ..., n(\gamma) - 1\}$*;*

(c) $\theta^\gamma(t_i) \in \{0, \pm\pi\}$ *for each* $i \in \{1, ..., n(\gamma) - 1\}$*;*

(d) $\theta^\gamma(\bar{t}) \in \{0, \pm\pi\}$, $(\bar{t} \in Dom(\gamma))$ *iff* $\bar{t} = t_i$ *for some* $i \in \{1, ...n(\gamma) - 1\}$.

Proof.

Proof that **(a)** implies **(c)**. By definition $\theta^\gamma(0) = 0$. Proposition 3.8 implies that the vectors $v^\gamma(G(\gamma(t_{i+1}), t_{i+1}; 0)$ and $v^\gamma(G(\gamma(t_i), t_i; 0)$ are parallel. Since $\theta^\gamma(t_{i+1})$ and $\theta^\gamma(t_i)$ measure precisely the angle between those vectors and $G(0)$, we have $\theta^\gamma(t_{i+1}) = \theta^\gamma(t_i) \pm \pi$.

Proof that **(c)** implies **(a)**. From **c)** we have $\theta^\gamma(t_1) \in \{0, \pm\pi\}$, then for some $b \in \mathbf{R}$ (that by genericity we may assume different from 0 and 1) it holds $v^\gamma(G(\gamma(t_1)), t_1; 0) = bG(\gamma(0))$. Now if we suppose that γ corresponds to constant control $+1$ in the interval $[0, t_1]$ (the opposite case being similar), we have $bG(\gamma(0)) = b(F+G)(\gamma(0)) = v^\gamma(b(F+G)(\gamma(t_1)), t_1; 0)$. From the injectivity of the map $v_0 \to v^\gamma(v_0, t_0; t_1)$ we obtain $\gamma(t_1) \in \Delta_A^{-1}(0)$. Using Proposition 3.3 it follows **(a)**.

Proof that **(b)** implies **(c)** and viceversa. Clearly **(b)** follows from **(c)**, let us prove the opposite. Let λ be the covector associated to γ. From **(b)** we have that $\lambda(0)$ is orthogonal to $G(0)$, hence γ switches iff $\theta^\gamma \in \{0, \pm\pi\}$. Thus **(c)** follows.

Proof that **(a)** implies **(d)**. It is a consequence of the generic condition $\theta^\gamma(\bar{t}) = 0 \Rightarrow \dot{\theta}^\gamma(\bar{t}) \neq 0$.

The implication **(d)**$\Rightarrow$**(c)** is obvious. This concludes the proof. □

PROPOSITION 3.10. *Let (γ, t_1) be a NTAE and $0 =: t_0 < t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. Suppose that for some $i \in \{0, 1, ..., n(\gamma) - 2\}$, $\Delta_A^{-1}(0)$ is inverse at the points $\gamma(t_i)$, $\gamma(t_{i+1})$. Then $\theta^\gamma(t_{i+1}) = \theta^\gamma(t_i)$.*

Proof. Set $(F + G)(\gamma(t_i)) = \alpha_i(F - G)(\gamma(t_i))$ and $F(\gamma(t_i)) = \beta_i G(\gamma(t_i))$. Under generic assumptions, α_i and β_i are well defined for each $i = 1, ...n(\gamma) - 1$ and it holds:

$$\alpha_0 = -1, \quad \beta_0 = 0 \text{ and } \alpha_i, \beta_i \notin \{0, \pm 1\}, \quad \beta_i = \frac{1 + \alpha_i}{1 - \alpha_i}, \quad i \in \{1, ..., n(\gamma) - 1\}.$$

Now if $\Delta_A^{-1}(0)$ is inverse at both points $\gamma(t_i), \gamma(t_{i+1})$ $(i \in \{0, ..., n(\gamma) - 2\}$ then $\alpha_i, \alpha_{i+1} < 0$ and we have $\beta_i, \beta_{i+1} \in]-1, 1[$ (see Figure 3.2). Recalling the definition of θ^γ, from Proposition 3.8 it follows the conclusion. □

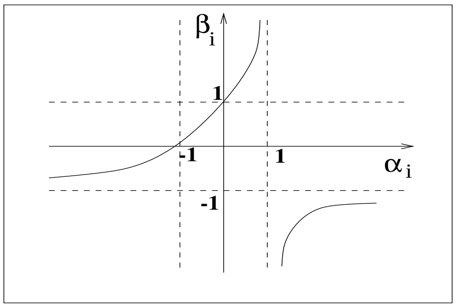


Fig. 3.2.

From Propositions 3.3 and 3.9, using the definitions of the times s_1^+, $s_1'^+$, t_f^+, s_1^-, $s_1'^-$, t_f^-, it follows:

COROLLARY 3.11. *Let γ be an extremal trajectory exiting the origin with control $+1$, then its first switching can occur on $\Delta_A^{-1}(0)$ only if $s_1 \neq 0$ (cond. A) or $s_1' \neq 0$ (cond. B) or $|\theta^+(t_f^+)| = \pi$ (cond. C). Moreover at most one of the conditions A, B, C holds and the corresponding time is the first switching time of γ and the first time at which γ^+ intersect $\Delta_A^{-1}(0)$. A similar result holds for γ^- and for the times s_1^-, $s_1'^-$, t_f^-.*

Referring to conditions 1, 2, 3 of Observation 1, conditions A and B correspond both to case 2 or case 1 (with $s_1^+ \neq 0$ or $s_1'^+ \neq 0$) and condition C corresponds to case 3. Moreover it is clear that for a NTAE (γ, t_1), t_1 is the first time at which γ reaches $\Delta_A^{-1}(0)$. In particular if the trajectory exits the origin with control $+1$ we have $t_1 = s_1^+$ or $t_1 = s_1'^+$ or $t_1 = t_f^+$. In the case $s_1^+ = 0$ and $s_1'^+$ not defined or viceversa and $|\theta^+(t_f^+)| < \pi$ (that implies $t_f^+ = \tau$) there are not NTAE exiting the origin with control $+1$ (an abnormal extremal exists but it never switches). This case corresponds to case 1 of Observation 1 with $s_1^+ = 0$ and s'^+ not defined or viceversa. These observations are collected in the following:

COROLLARY 3.12. *There are at most two maximal NTAE. Moreover:*

(♠) *one exits the origin with control $+1$ and its first switching is at*
- s_1^+ *iff* $s_1^+ \neq 0$;
- $s_1'^+$ *iff* $s_1'^+ \neq 0$;
- t_f^+ *iff* $|\theta^+(t_f^+)| = \pi$;

(♣) *the other exits the origin with control -1 and its first switching is at*
- s_1^- *iff* $s_1^- \neq 0$;
- $s_1'^-$ *iff* $s_1'^- \neq 0$;
- t_f^- *iff* $|\theta^-(t_f^-)| = \pi$.

Finally if $|\theta^\pm(t_f^\pm)| = \pi$ then $\Delta_A^{-1}(0)$ is direct at $\gamma(t_f^\pm)$.

The following two Propositions describe the position of the switching curves of the strips whose borders are abnormal extremals.

PROPOSITION 3.13. *Let (γ, t_1) be a NTAE, t_i and t_{i+1} two consecutive switching times, $\mathcal{S}$ a strip such that $\gamma \in \partial S$ and U^i, U^{i+1} two sufficiently small neighborhoods of $\gamma(t_i)$ and $(\gamma(t_{i+1}))$. Moreover let U^i_{in} and U^i_{out} (resp. U^{i+1}_{in}, U^{i+1}_{out}) be the two connected components of $U^i \setminus \Delta_A^{-1}(0)$ (resp. $U^{i+1} \setminus \Delta_A^{-1}(0)$) chosen in such a way that γ enters U^i_{in} (resp. U^{i+1}_{in}). Under generic conditions we have the following cases:*

(1) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.*
In this case if the switching locus of $\mathcal{S}$ passing through $\gamma(t_i)$ lies in U^i_{in} (resp. U^i_{out}) then the switching locus of $\mathcal{S}$ passing through $\gamma(t_{i+1})$ lies in U^{i+1}_{out} (resp. U^{i+1}_{in}).

(2) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.*
In this case if the switching locus of $\mathcal{S}$ passing through $\gamma(t_i)$ lies in U^i_{in} (resp. U^i_{out}) then the switching locus of $\mathcal{S}$ passing through $\gamma(t_{i+1})$ lies in U^{i+1}_{in} (resp. U^{i+1}_{out}).

(3) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.*
In this case we have the same conclusion as in case **(2)**.

(4) *$\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.*
In this case we have the same conclusion as in case **(1)**.

Proof. Let f_i (resp. A_i) be the sign of $-\Delta_B/\Delta_A$ (resp. Δ_A) on U^i_{in} and B_i the sign of Δ_B on U^i. By hypothesis, taking U^i sufficiently small, these quantities are well defined and we have $f_i = -A_iB_i$. Moreover set $\theta_i = +1$ if $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$

and $\theta_i = -1$ if $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$.

Claim $B_{i+1} = \theta_i B_i$

Proof of the Claim. From $sgn(\dot\theta^\gamma(t)) = sgn(\Delta_B(\gamma(t)))$ we have that $\bar v^\gamma(t)$ (see formula (2.7)) is a vector rotating counterclockwise in U^i (resp. U^{i+1}) iff $B_i > 0$ (resp. $B_{i+1} > 0$). Recalling that $\theta^\gamma(t_i), \theta^\gamma(t_{i+1}) \in \{0, \pm\pi\}$ it is clear that B_i and B_{i+1} have the same sign iff $\theta^\gamma(t_{i+1}) = \theta^\gamma(t_i) \pm \pi$. The Claim is proved.

Case 1 First suppose Δ_A direct at $\gamma(t_i)$. In this case from Proposition 3.3 we clearly have $A_{i+1} = -A_i$. Now if the switching loci of $\mathcal{S}$ lie one in U^i_{in} and one in U^{i+1}_{in} (resp. U^i_{out} and U^{i+1}_{out}) then, from Lemma 2.8, we have $f_i = -f_{i+1}$. This occurs iff $-A_iB_i = +A_{i+1}B_{i+1} = -A_iB_i\theta_i$, from which it follows $\theta_i = +1$.

On the other side if the switching loci of $\mathcal{S}$ lie one in U^i_{in} and one in U^{i+1}_{out} (resp. U^i_{out} and U^{i+1}_{in}) then $f_i = +f_{i+1}$ that occurs iff $\theta_i = -1$.

Case 2 If Δ_A is inverse at $\gamma(t_i)$ we have $A_{i+1} = +A_i$. Now if the switching loci of $\mathcal{S}$ lie one in U^i_{in} and one in U^{i+1}_{in} (resp. U^i_{out} and U^{i+1}_{out}) we have $f_i = -f_{i+1}$. This occurs iff $\theta_i = -1$.

On the other side if the switching loci of $\mathcal{S}$ lie one in U^i_{in} and U^{i+1}_{out} (resp. U^i_{out} and U^{i+1}_{in}) then $f_i = +f_{i+1}$ that occurs iff $\theta_i = +1$. □

PROPOSITION 3.14. *Let (γ, t_1) be a NTAE and let $\mathcal{S}^1$ and $\mathcal{S}^2$ be two strips such that $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$. Let $\bar t$ be a switching time for γ and U a small neighborhood of $\gamma(\bar t)$ such that $U \setminus \Delta_A^{-1}(0)$ has two connected components U_{in} and U_{out}, chosen in such a way γ enters U from U_{in}. Then, under generic conditions, the switching loci of $\mathcal{S}^1$ and $\mathcal{S}^2$ passing through $\gamma(\bar t)$ satisfy the following:*

(a) *they lie both in U_{in} or both in U_{out};*

(b) *they are tangent to supp(γ) in $\gamma(\bar t)$.*

Proof of (a). By the analysis of the singularities at the first switching time (see Figure 4.1) we know that **(a)** is true in the special case $\bar t = t_1$. Using Proposition 3.13 and by induction it follows the thesis. □

Proof of (b). We prove **(b)** by induction on the switching time. We start considering the first switching time, and treat only the case in which γ exits the origin with constant control $+1$, the opposite case being similar. First suppose that $\bar t$ is the first switching time for γ and assume $s_1^+ > 0$ (i.e. $\bar t = s_1^+$), the case $s_1'^+ > 0$ and $|\theta^+(t_f^+)| = \pi$ being similar. Set $x = (x_1, x_2)$ and chose a local system of coordinates in such a way that:

$$Y \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad X(x) = \begin{pmatrix} -1 + a_1x_1 + a_2x_2 + O(|x|^2) \\ b_1x_1 + b_2x_2 + O(|x|^2) \end{pmatrix},$$

$$X(x) = \begin{pmatrix} c_0 + c_1(x_1 - s_1^+) + c_2x_2 + O(|x - (s_1^+, 0)|^2) \\ d_1(x_1 - s_1^+) + d_2x_2 + O(|x - (s_1^+, 0)|^2) \end{pmatrix}.$$

Generically $\Delta_B(\gamma^+(s_1^+)) \neq 0$ and thus a switching curve generates at $\gamma^+(s_1^+)$.

Let $(\bar\gamma, \bar\lambda)$ be an extremal trajectory of (2.1) corresponding to constant control -1 in the interval $[0, \tau_1[$ and to constant control $+1$ in the interval $[\tau_1, \tau_2]$. We consider the trajectories $\bar\gamma$ that are near to γ, that is those corresponding to (τ_1, τ_2) in a neighborhood of $(0, s_1^+)$. We have:

$$\bar\gamma(\tau_1) = \begin{pmatrix} -\tau_1 + O(\tau_1^2) \\ -\frac{1}{2}b_1\tau_1^2 + O(\tau_1^3) \end{pmatrix}, \quad \bar\gamma(\tau_2) = \begin{pmatrix} -\tau_1 + \tau_2 + O(\tau_1^2) \\ -\frac{1}{2}b_1\tau_1^2 + O(\tau_1^3) \end{pmatrix}.$$

Now if τ_1, τ_2 are switching times for $\bar\gamma$ we must have:

$$\bar\lambda(\tau_1) \cdot G(\bar\gamma(\tau_1)) = 0, \tag{3.1}$$

$$\bar{\lambda}(\tau_2) \cdot G(\bar{\gamma}(\tau_2)) = 0. \tag{3.2}$$

Moreover from equation **i)** of the PMP we have $\bar{\lambda}(\tau_1) = \bar{\lambda}(\tau_2) =: \bar{\lambda}$. Finally set $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)$ and normalize $\bar{\lambda}$ in such a way $\bar{\lambda}_2 = \sqrt{1-\bar{\lambda}_1}$. Equations (3.1), (3.2) became:

$$\begin{aligned}
f_1(\lambda_1, \tau_1, \tau_2) &:= \bar{\lambda}_1(1 + a_1\tau_1) + \sqrt{1-\bar{\lambda}_1}(\frac{1}{2}b_1\tau_1) + O(\tau_1^2) = 0, \\
f_2(\lambda_1, \tau_1, \tau_2) &:= \bar{\lambda}_1(\frac{1}{2}(1-c_0) - c_1(-\tau_1 + (\tau_2 - s_1^+))) + \\
&\quad + \sqrt{1-\bar{\lambda}_1}(-\frac{1}{2}d_1(-\tau_1 + (\tau_2 - s_1^+))) + O(\tau_1^2) + O((\tau_2 - s_1^+)^2) = 0.
\end{aligned}$$

These are two equations for the variable $(\bar{\lambda}_1, \tau_1, \tau_2)$ and $(0, 0, s_1^+)$ is a solution. We compute the 2×2 Jacobian matrix of partial derivatives of f_1 and f_2 with respect to (λ_1, λ_2) and check that its determinant at the point $(0, 0, s_1^+)$ is equal to $-d_1/2$. Under the generic assumption $d_1 \neq 0$, we can solve the system in a neighborhood of $(0, 0, s_1^+)$ expressing $(\bar{\lambda}_1, \tau_2)$ as a function of τ_1. This yields:

$$\left.\frac{\partial \tau_2}{\partial \tau_1}\right|_{(0,0,s_1^+)} = -\frac{(b_1(1-c_0) - 2d_1)}{2d_1} =: m, \tag{3.3}$$

hence $\tau_2 = s_1^+ + m\tau_1 + O(\tau_1^2)$ and under the generic condition $m \neq 1$ the parametric expression for the switching curve starting at $(s_1^+, 0)$ is:

$$\begin{aligned}
x_1(\tau_1) &= s_1^+ + (m-1)\tau_1 + O(\tau_1^2), \\
x_2(\tau_1) &= -\frac{1}{2}b_1\tau_1^2 + O(\tau_1^3).
\end{aligned}$$

If $m \neq 0$, we can use τ_2 as parameter so:

$$\begin{aligned}
x_1(\tau_2) &= s_1^+ + \frac{m-1}{m}(\tau_2 - s_1^+) + O((\tau_2 - s_1^+)^2), \\
x_2(\tau_2) &= -\frac{b_1}{2m^2}(\tau_2 - s_1^+)^2 + O((\tau_2 - s_1^+)^3).
\end{aligned}$$

Assuming the generic condition $c_0 \neq 1$ and $b_1 \neq 0$, we may express x_2 as a function of x_1:

$$x_2 = -\frac{2d_1^2}{b_1(1-c_0)^2}(x_1 - s_1^+)^2 + O((x_1 - s_1^+)^3). \tag{3.4}$$

Notice that this curve can be of kind C or $\bar{C}$. This proves that the switching curve starting at the first switching time is tangent to $Supp(\gamma)$.

Now we have to prove the induction step, that is if t_n, t_{n+1} are two consequent switching times for γ, and the switching curve passing through t_n is tangent to γ then the switching curve passing through t_{n+1} is tangent to γ as well. Let us consider only the case in which γ corresponds to constant control $+1$ on $[t_n, t_{n+1}]$, the opposite case being similar. Choose a local system of coordinates and rescale the time in such a way that:

$$t_n = 0, \quad \gamma(t_n) = 0,$$

$$Y \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad X(x) = \begin{pmatrix} e_0 + e_1 x_1 + e_2 x_2 + O(|x|^2) \\ f_1 x_1 + f_2 x_2 + O(|x|^2) \end{pmatrix},$$

$$X(x) = \begin{pmatrix} g_0 + g_1(x_1 - t_{n+1}) + g_2 x_2 + O(|x - (t_{n+1}, 0)|^2) \\ h_1(x_1 - t_{n+1}) + h_2 x_2 + O(|x - (t_{n+1}, 0)|^2) \end{pmatrix}.$$

We can parameterize the switching curve C_n through $\gamma(t_n)$ in the following way. Given $x \in C_n$ there exists (γ_x, λ_x) extremal that switches at x at time $\tau_n(x)$. By induction we may assume that τ_n is invertible for x near $\gamma(t_n)$ and parameterize C_n by τ_n:

$$x_1(\tau_n) = \alpha_n \tau_n + O(\tau_n^2),$$
$$x_2(\tau_n) = \beta_n \tau_n^2 + O(\tau_n^3),$$

for some α_n, β_n that by genericity we may assume different from zero.
Let $(\tilde\gamma, \tilde\lambda) := (\tilde\gamma, \tilde\lambda)_{\tau_n}$ be the extremal trajectory of (2.1) switching at τ_n on C_n and let τ_{n+1} be the next switching time. We have:

$$\tilde\gamma(\tau_n) = \begin{pmatrix} \alpha_n \tau_n + O(\tau_n^2) \\ \beta_n \tau_n^2 + O(\tau_n^3) \end{pmatrix}, \quad \tilde\gamma(\tau_{n+1}) = \begin{pmatrix} \alpha_n \tau_n + \tau_{n+1} + O(\tau_n^2) \\ \beta_n \tau_n^2 + O(\tau_n^3) \end{pmatrix}.$$

and:

$$\tilde\lambda(\tau_n) \cdot G(\tilde\gamma(\tau_n)) = 0, \tag{3.5}$$
$$\tilde\lambda(\tau_{n+1}) \cdot G(\tilde\gamma(\tau_{n+1})) = 0. \tag{3.6}$$

With computations entirely similar to the previous ones and using similar generic conditions we conclude that the switching curve passing through $\gamma(t_{n+1})$ has the expression:

$$x_1(\tau_{n+1}) = t_{n+1} + \alpha_{n+1}(\tau_{n+1} - t_{n+1}) + O((\tau_{n+1} - t_{n+1})^2),$$
$$x_2(\tau_{n+1}) = \beta_{n+1}(\tau_{n+1} - t_{n+1})^2 + O((\tau_{n+1} - t_{n+1})^3),$$

for some $\alpha_{n+1}, \beta_{n+1} \neq 0$. This conclude the proof. □

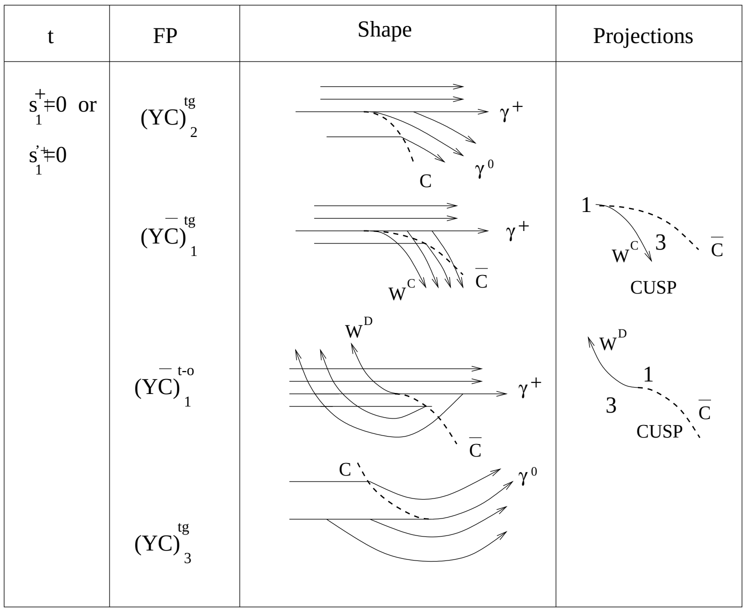


Fig. 4.1

4. Singularities. In this Section we describe all possible Frame Points occurring along a NTAE.

We start to describe the first singularity for the NTAE exiting the origin with control $+1$, the opposite case being similar. We refer to Figure 4.1 where all extremal trajectories are depicted in a neighborhood of the Frame Points. Following Corollary 3.12 a NTAE generates at time s_1^+ (iff $s_1^+ \neq 0$), or at time $s_1'^+$ (iff $s_1'^+ \neq 0$), or at time t_f^+ (iff $|\theta^+(t_f^+)| = \pi$). Assume $s_1^+ \neq 0$ then a switching curve tangent to $Supp(\gamma^+)$ bifurcates from $\gamma^+(s_1^+)$. Recall formula (3.4) of the proof of **b)** of Proposition 3.14. Using the definition of θ^+ and s_1^+, as in Proposition 3.1 of [7], one gets that $b_1 > 0,\ d_1 < 0$ (we would have $b_1 < 0,\ d_1 > 0$, if $s_1^+ \neq 0$) and $c_0 < 1$. Hence the switching curve is bifurcating to the right of $Supp(\gamma^+)$. Moreover it follows that m, defined in formula (3.3), is bigger than 1 and being $t_1 > 0$ we have (in formula (3.4)) $x_1 > s_1^+$. If the switching curve is of kind C we call the singularity $(Y,C)_2^{tg}$. If the switching curve is of kind $\bar{C}$ and $\Delta_A^{-1}(0)$ is direct at $\gamma^+(s_1^+)$ we call the singularity $(Y,\bar{C})_1^{tg}$. Finally if the switching curve is of kind $\bar{C}$ and $\Delta_A^{-1}(0)$ is inverse at $\gamma^+(s_1^+)$ we call the singularity $(Y,\bar{C})_1^{t-o}$. These name are chosen in accordance with [5, 16]. The case in which $s_1'^+ \neq 0$ is entirely similar.

The case in which the NTAE starts at $\gamma(t_f^+)$ (that happens iff $|\theta^+(t_f^+)| = \pi$), is again described by formula (3.4) with s_1^+ replaced by t_f^+. In this case, reasoning as in [7], we get $c_0 > 1$ and $b_1 d_1 < 0$. It follows that the switching curve bifurcates to the right, $m < 1$, and in formula (3.4) we have $x_1 < t_f^+$. Moreover, at t_f^+, γ^+ stops to be extremal and $\Delta_A^{-1}(0)$ is direct at $\gamma(t_f^+)$ (see Proposition 3.10). We call this singularity $(Y,C)_3^{tg}$.

To classify the other generic singularities involving a NTAE we consider at the Frame Points:

- $\Delta_A^{-1}(0)$ direct or inverse;
- switching in U_{in} or U_{out} according to Proposition 3.13;
- all the *essentially different* directions of the exiting abnormal trajectory.

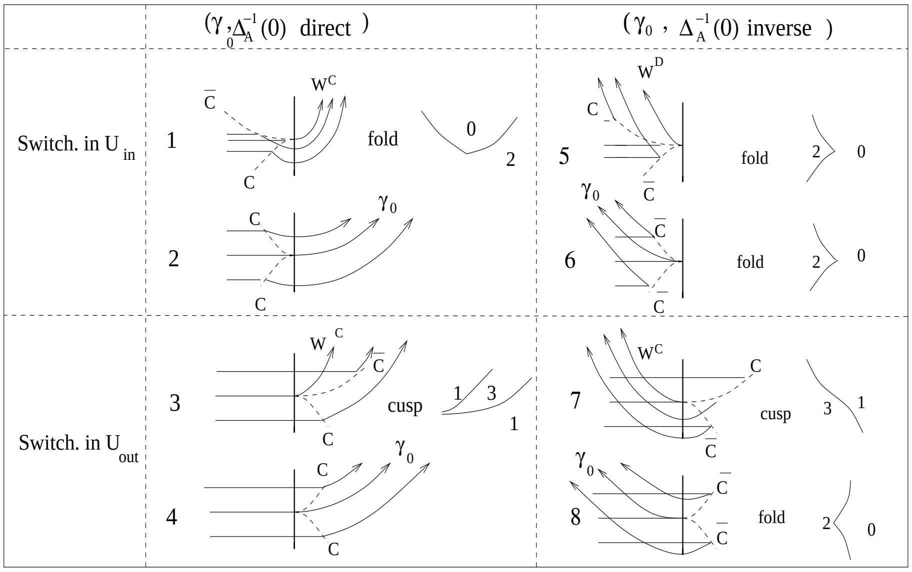


Fig. 4.2

We obtain 24 types of singularities. The singularities with entering abnormal extremals of kind γ_0 are showed in Figure 4.2, while in Figure 4.3 all the possible singularities for a NTAE of the kind W^C, and W^D are listed. In these figures we also indicate the labels *fold, cusp, bifold or ribbon* in accordance with Definition 2.9.

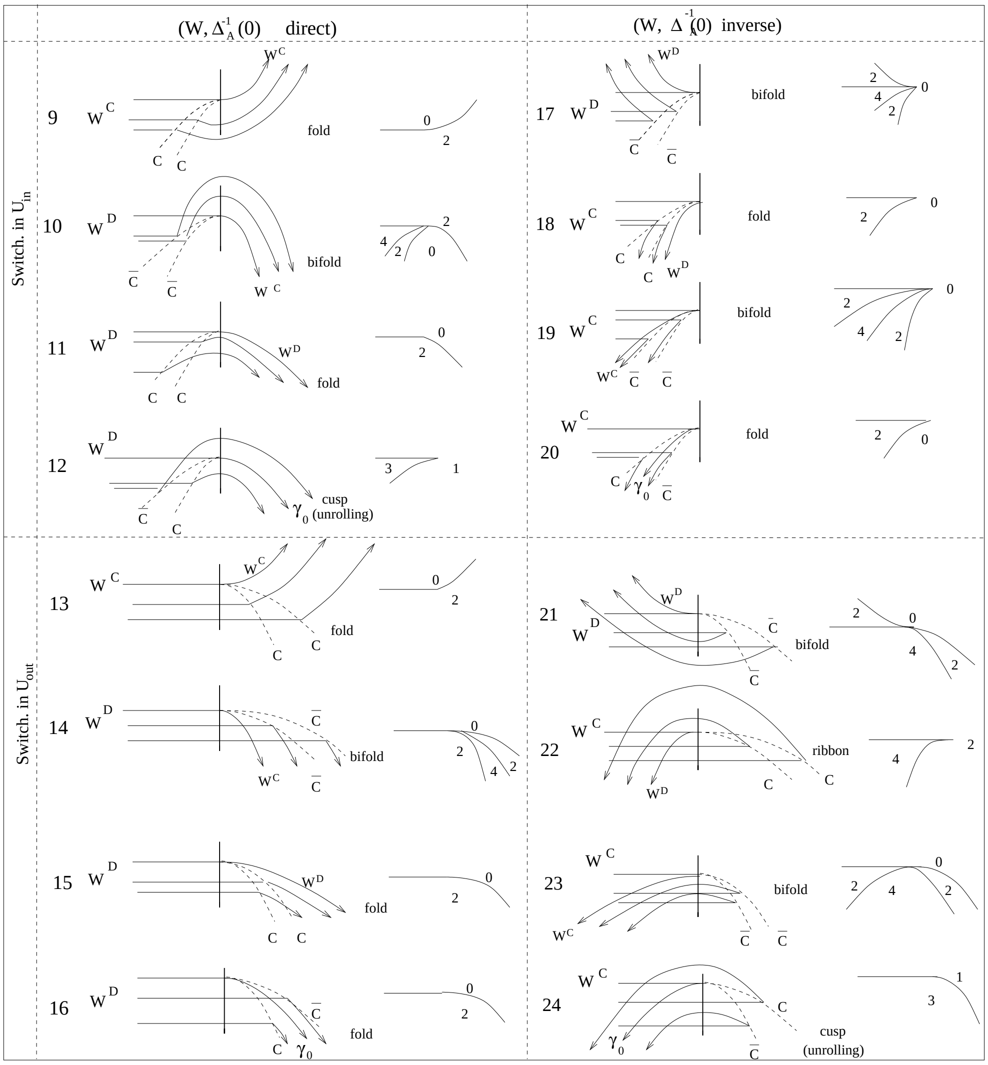


Fig. 4.3

5. Classification of Abnormal Extremals. In this section we prove Theorem 2.2.

From now on we fix a positive time $\tau > 0$. Using Corollary 3.12, we have that for each system $(F, G) \in \Xi$ there exists exactly two maximal abnormal extremals $\gamma_A^{\pm} = \gamma_A^{\pm}((F, G), \tau)$ in time τ exiting the origin respectively with control ± 1.

Definition 5.1. *We say that a set A provides a generic classification of abnormal extremals (in time τ) if there exists a generic subset Π of Ξ and a map $\Phi : \Pi \to A$ such that $\Phi((F, G)) = \Phi((F', G'))$, $(F, G), (F', G') \in \Pi$, if and only if the corresponding maximal abnormal extremals present the same finite sequence of generic singularities.*

Remark. From Section 4, we know the structure of the set of extremal trajectories near each generic singularity. Hence, if two abnormal extremals present the same sequence of singularities then the synthesis near them is exactly the same.

We provide a generic classification through a set of words A recognizable by an *automaton*. Therefore, proving Theorem 2.2 amounts precisely to constructing an automaton describing all possible sequences of generic singularities along a NTAE.

First we build an automaton, naturally associated to a system, with the simplest possible set of edges. By this automaton we can prove that the ribbon and the bifold singularities are realized, but more than one sequence of singularities may correspond to a recognizable word. Then we build a more complicated automaton that has the required property, i.e. to every recognizable word it corresponds one and only one sequence of generic singularities.

Let us first recall some definition from automata theory. For a more extensive and detailed treatment of the subject we refer to [10].

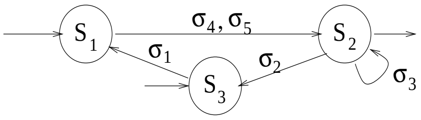
Definition 5.2. *Let Σ be a finite set and consider the set Σ^* of ordered n–tuples $s = (\sigma_1, ..., \sigma_k)$, $\sigma_i \in \Sigma$ $(i = 1, ...k)$, $k \geq 0$. We call Σ the* alphabet, *$\sigma \in \Sigma$ a* letter, *$s = (\sigma_1, ..., \sigma_k) \in \Sigma^*$ a* word of length *k and Σ^* the set of words generated by Σ.*

The set of words generated by an alphabet is a set with a simple structure and a classification based on such a set is quite satisfying. Such a kind of classification was given, for example, for the sequence of generic singularities along $\gamma^{\pm}$ in [14]. For abnormal extremals we have to use a set with a more complicate structure.

Definition 5.3. *Let Σ be a finite* alphabet, *an* automaton *$\mathcal{A}$ over Σ consists of the following:*

- *a finite set* $\mathbf{S}$ *whose elements are called* states;
- *a set of* initial states $\mathbf{I} \subseteq \mathbf{S}$;
- *a set of* terminal states $\mathbf{T} \subseteq \mathbf{S}$;
- *a set of* edges *that is a subset* $\mathbf{E} \subseteq \mathbf{S} \times \Sigma \times \mathbf{S}$. *An* edge *is indicated as (S_1, σ, S_2) and we say that it begins at S_1, it ends at S_2 and it carries the label σ.*

Usually an *automaton* is represented by a set of circles (states) and a set of arrows that connect the circles (the edges). The initial (resp. final) states are labelled by arrows pointing towards (resp. away from) the circle. If there are several edges beginning and ending at the same states they are replaced by a single arrow carrying several labels.



A *path* in $\mathcal{A}$ is a finite sequence of edges of the type $(S_1, \sigma_1, S_2)(S_2, \sigma_2, S_3)...(S_k, \sigma_k, S_{k+1})$. If $S_1 \in \mathbf{I}$ and $S_{k+1} \in \mathbf{T}$ we say that the *path* is successful.

Definition 5.4. *A set of words $\Omega \subset \Sigma^*$ is said to be recognizable by $\mathcal{A}$ if for every word $(\sigma_1, \sigma_2, ..., \sigma_m) \in \Omega$ of length m there exists $S_1, ..., S_{m+1} \in \mathbf{S}$ such that:*

- $(S_i, \sigma_i, S_{i+1}) \in \mathbf{E}$ *for every* $i = 1, ..., m$;
- $(S_1, \sigma_1, S_2)(S_2, \sigma_2, S_3)...(S_m, \sigma_m, S_{m+1})$ *is a successful path.*

The set of words recognizable by an automaton share some regularity properties, in particular they are studied in automata theory (see [10]). Let us resume all the information on the switching strategy of an abnormal extremal via three rules. Let (γ, t_1) be a NTAE and $t_1 < t_2 ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of its switching times.

R1. Let $\mathcal{S}^1$ and $\mathcal{S}^2$ be the two strips such that $\gamma \in \mathcal{S}^1 \cap \mathcal{S}^2$ and let $\bar{\mathcal{S}} := \mathcal{S}^1 \cup \mathcal{S}^2$. We have the following cases:

(1) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.
In this case if the switching locus of $\bar{\mathcal{S}}$ passing through $\gamma(t_i)$ lies in U^i_{in} (resp. U^i_{out}) then the switching locus of $\bar{\mathcal{S}}$ passing through $\gamma(t_{i+1})$ lies in U^{i+1}_{out} (resp. U^{i+1}_{in}).

(2) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.
In this case if the switching locus of $\bar{\mathcal{S}}$ passing through $\gamma(t_i)$ lies in U^i_{in} (resp. U^i_{out}) then the switching locus of $\bar{\mathcal{S}}$ passing through $\gamma(t_{i+1})$ lies in U^{i+1}_{in} (resp. U^{i+1}_{out}).

(3) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.
In this case we have the same conclusion as in case **(2)**.

(4) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.
In this case we have the same conclusion as in case **(1)**.

The rule **R1.** is a direct consequence of Propositions 3.13 and 3.14 **a)**.
R2. If $\Delta_A^{-1}(0)$ is inverse at t_i and t_{i+1} then $\theta^\gamma(t_{i+1}) = \theta^\gamma(t_i)$.
The rule **R2.** follows from Proposition 3.10.
R3. Only the following consecutive singularities are possible:

singularity at t_i	possible singularity at t_{i+1}	AA(i)
$(YC)_2^{tg}$,$(YC)_3^{tg}$,2,4,6,8,12,16,20,24	1,2,3,4,5,6,7,8	γ^0
$(Y\bar{C})_1^{tg}$,1,3,7,9,10,13,14,19,23	9,13,18,19,20,22,23,24	W^C
$(Y\bar{C})_1^{t-o}$,5,11,15,17,18,21,22	10,11,12,14,15,16,17,21	W^D

Clearly, if an abnormal arc of kind γ^0 (resp. W^C, W^D) exits the singularity $\gamma(t_i)$ ($i = 1, ..., n(\gamma) - 1$), then an abnormal arc of kind γ^0 (resp. W^C, W^D) enters the singularity $\gamma(t_{i+1})$. Thus **R3.** can be directly checked using Figure 4.1, 4.2 and 4.3.

We now are ready to build an automaton $\mathcal{A}$. For us the set of *states* is the set of the 28 singularities:

$$\mathbf{S} := \{(YC)_2^{tg}, (Y\bar{C})_1^{tg}, (Y\bar{C})_1^{t-o}, (YC)_3^{tg}, \\ 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24\}$$

and the alphabet is:

$$\Sigma := \{0, \pi\}$$

that is (if we are considering two singularities at t_i and t_{i+1}) the set of values assumed by the function $\Delta\theta_i^\gamma := |\theta^\gamma(t_{i+1}) - \theta^\gamma(t_i)|$. The set of *initial states* is constituted by the singularities $(YC)_2^{tg}$, $(Y\bar{C})_1^{tg}$, $(Y\bar{C})_1^{t-o}$, $(YC)_3^{tg}$ and the set of *terminal states* coincides with $\mathbf{S}$. Using rules **R1**$\div$ **R3** we obtain Table A that show how the *edges* connect the *states*, that is it describes the set of edges $\mathbf{E}$. For example from the *state* (singularity) 18, using the *letter* π we may reach the states $14, 15, 16$. This means that the edges of $\mathbf{E}$ with label π, that start at the state 18 are: $(18, \pi, 14), (18, \pi, 15), (18, \pi, 16)$. It is clear that for this automaton every word of Σ^* is recognizable, but $\mathcal{A}$ does not provide a generic classification because a word corresponds to more than one sequence of singularities. However it describes in a simple way the set of abnormal extremals and, in particular, from Table A we have:

THEOREM 5.5. *All states $1 \div 24$ can be reached with at most two edges. More precisely only the singularity 17 needs in fact two edges. Moreover, the singularity number 22 (the ribbon) can be realized with the edge $((Y\bar{C})_1^{tg}, \pi, 22)$ and the singularity number 10 (that is a bifold) with the edge $((Y\bar{C})_1^{t-o}, \pi, 10)$.*

letters→ *states* ↓	0	π
$(YC)_2^{tg}$	1,2,5,6	3,4,7,8
$(Y\bar{C})_1^{tg}$	9,18,19,20	13,22,23,24
$(Y\bar{C})_1^{t-o}$	14,15,16,21	10,11,12
$(YC)_3^{tg}$	1,2,5,6	3,4,7,8
1	13,22,23,24	9,18,19,20
2	3,4,7,8	1,2,5,6
3	9,18,19,20	13,22,23,24
4	1,2,5,6	3,4,7,8
5	10,11,12,17	14,15,16
6	1,2,5,6	3,4
7	13,22,23,24	9
8	3,4,7,8	1,2
9	13,22,23,24	9,18,19,20
10	13,22,23,24	9,18,19,20
11	14,15,16,21	10,11,12,17
12	3,4,7,8	1,2,5,6
13	9,18,19,20	13,22,23,24
14	9,18,19,20	13,22,23,24
15	10,11,12,17	14,15,16,21
16	1,2,5,6	3,4,7,8
17	10,11,12,17	14,15,16
18	10,11,12,17	14,15,16
19	9,18,19,20	13
20	1,2,5,6	3,4
21	14,15,16,21	10,11,12
22	14,15,16,21	10,11,12
23	13,22,23,24	9
24	3,4,7,8	1,2

Table A

In Figure 5.1 it is shown the automaton with all states, but only the edges of the kind $((YC)_2^{tg},\ .\ ,\ .\ ),\ ((Y\bar{C})_1^{tg},\ .\ ,\ .\ ),\ (15,\ .\ ,\ .\ )$.

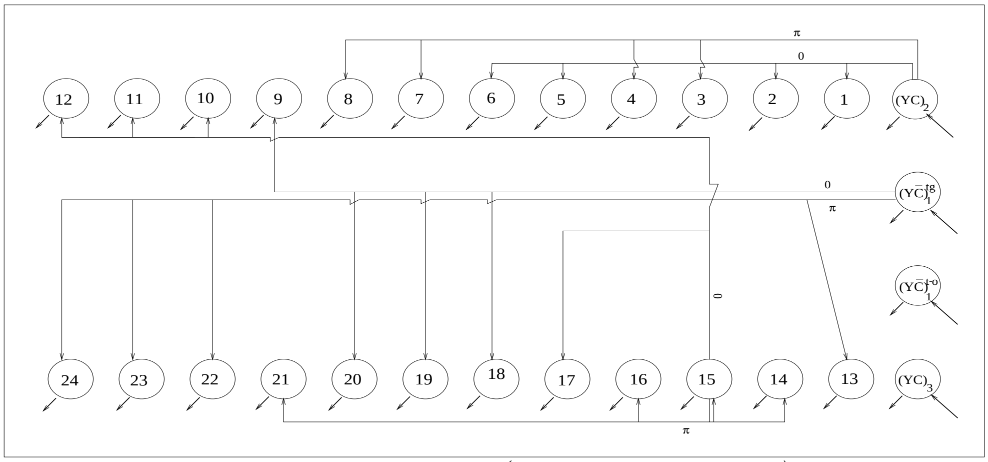


Fig. 5.1. *The automaton (some edges are omitted)*

To build a new automaton $\mathcal{A}'$ that provides a generic classification, we need to include more information in the alphabet. First of all we assign a label to the entry arrows I_1, I_2, I_3, I_4, corresponding respectively to the singularities $(YC)_2^{tg}$, $(Y\bar{C})_1^{tg}$, $(Y\bar{C})_1^{t-o}$,

$(YC)_3^{tg}$. Then we introduce more information in the letters, i.e. we need a bigger alphabet. To do this, given a generic singularity on a NTAE, we want to include the following data relatively to the subsequent singularity:

- $\Delta_A^{-1}(0)$ direct or inverse (indicated by D and I respectively),
- the kind of exiting abnormal arc (i.e. γ^0, W^C or W^D).

In this way, the automaton $\mathcal{A}'$ is formed by $\mathbf{S}' = \mathbf{S}$, $\Sigma' = \{0,\pi\} \times \{D,I\} \times \{\gamma^0, W^C, W^D\} \cup \Sigma'_1$ where $\Sigma'_1 = \{I_1, I_2, I_3, I_4\}$ and set of edges $\mathbf{E}'$. Every element of Σ' (that is not in Σ'_1) is indicated by a triplet (. , . , .). In Table B the set of edges $\mathbf{E}'$ (with labels not in Σ'_1) is completely described. Notice that not all words of $(\Sigma')^*$ are recognizable by $\mathcal{A}'$. For example the word $I_2(\pi, I, W^D)(0, D, W^C)$ is recognizable and it corresponds to the sequence of singularities $(Y\bar{C})_1^{tg} \to 22 \to 14$ while the word $I_1(0, I, W^C)$ is not recognizable.

With these definitions, to every word recognizable by $\mathcal{A}'$ it corresponds one and only one possible sequence of generic singularities along a NTAE. Theorem 2.2 is therefore proved.

letters→ *states* ↓	$(0,D,\gamma^0)$	$(0,D,W^C)$	$(0,D,W^D)$	$(0,I,\gamma^0)$	$(0,I,W^C)$	$(0,I,W^D)$
$(YC)_2^{tg}$	2	1	–	6	–	5
$(Y\bar{C})_1^{tg}$	–	9	–	20	19	18
$(Y\bar{C})_1^{t-o}$	16	14	15	–	–	21
$(YC)_3^{tg}$	2	1	–	6	–	5
1	–	13	–	24	23	22
2	4	3	–	8	7	–
3	–	9	–	20	19	18
4	2	1	–	6	–	5
5	12	10	11	–	–	17
6	2	1	–	6	–	5
7	–	13	–	24	23	22
8	4	3	–	8	7	–
9	–	13	–	24	23	22
10	–	13	–	24	23	22
11	16	14	15	–	–	21
12	4	3	–	8	7	–
13	–	9	–	20	19	18
14	–	9	–	20	19	18
15	12	10	11	–	–	17
16	2	1	–	6	–	5
17	12	10	11	–	–	17
18	12	10	11	–	–	17
19	–	9	–	20	19	18
20	2	1	–	6	–	5
21	16	14	15	–	–	21
22	16	14	15	–	–	21
23	–	13	–	24	23	22
24	4	3	–	8	7	–

Table B (first part)

letters→ *states* ↓	(π, D, γ^0)	(π, D, W^C)	(π, D, W^D)	(π, I, γ^0)	(π, I, W^C)	(π, I, W^D)
$(YC)_2^{tg}$	4	3	–	8	–	7
$(Y\bar{C})_1^{tg}$	–	13	–	24	23	22
$(Y\bar{C})_1^{t-o}$	12	10	11	–	–	–
$(YC)_3^{tg}$	4	3	–	8	7	–
1	–	9	–	20	19	18
2	2	1	–	6	–	5
3	–	13	–	24	23	22
4	4	3	–	8	7	–
5	16	14	15	–	–	–
6	4	3	–	–	–	–
7	–	9	–	–	–	–
8	2	1	–	–	–	–
9	–	9	–	20	19	18
10	–	9	–	20	19	18
11	12	10	11	–	–	17
12	2	1	–	6	–	5
13	–	13	–	24	23	22
14	–	13	–	24	23	22
15	16	14	15	–	–	21
16	4	3	–	8	7	–
17	16	14	15	–	–	–
18	16	14	15	–	–	–
19	–	–	–	–	13	–
20	4	3	–	–	–	–
21	12	10	11	–	–	–
22	12	10	11	–	–	–
23	–	9	–	–	–	–
24	2	1	–	–	–	–

Table B (second part)

We refer to Figure 5.2 for a graphic example of synthesis involving a *ribbon* singularity.

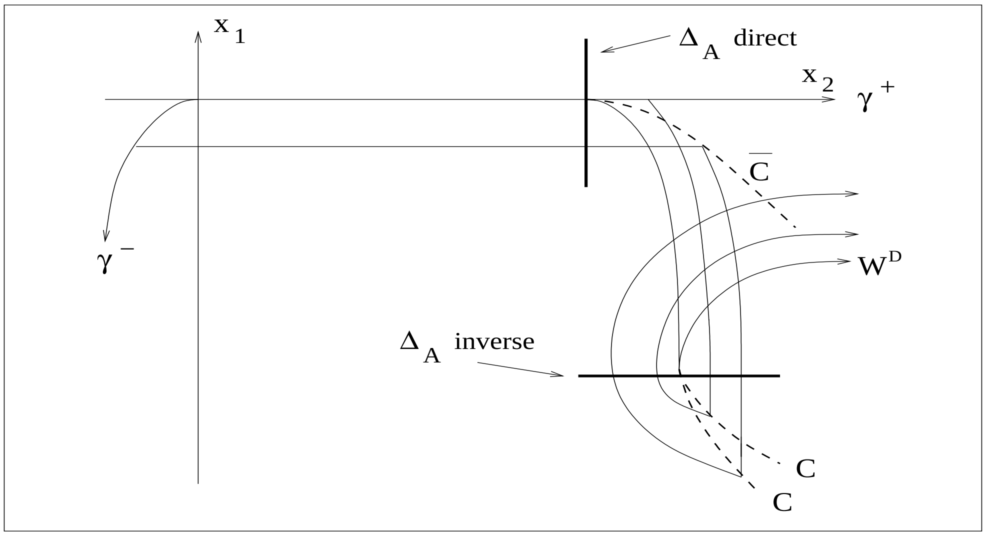


Fig. 5.2. *An example of synthesis involving a* ribbon

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